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The influence of vortex shedding on the generation of sound by convected turbulence

By M. S. HOWE
Engineering Department, University of Cambridge

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This paper discusses the theory of the generation of sound which occurs when a frozen turbulent eddy is convected in a mean flow past an airfoil or a semi-infinite plate, with and without the application of a Kutta condition and with and without the presence of a mean vortex sheet in the wake. A sequence of two-dimensional mathematical problems involving a prototype eddy in the form of a line vortex is examined, it being argued that this constitutes the simplest realistic model. Important effects of convection are deduced which hitherto have not been revealed by analyses which assume quadrupole sources to be at rest relative to the plate or airfoil. It is concluded that, to the order of approximation to which the sound from convected turbulence near a scattering body is usually estimated, the imposition of a Kutta condition at the trailing edge leads to a complete cancellation of the sound generated when frozen turbulence convects past a semi-infinite plate, and to the cancellation of the diffraction field produced by the trailing edge in the case of an airfoil of compact chord.

1. Introduction

This paper is concerned with the effect of vortex shedding on the efficiency of sound generation by convected turbulence. It is known (Ffowcs Williams & Hall 1970; Crighton & Leppington 1970) that the intensity of aerodynamic noise is greatly enhanced when the sound-producing turbulent quadrupoles (Lighthill 1952) are located near non-uniformities in the flow, such as the sharp edges of a strut or splitter plate. In the presence of a large-scale mean flow there is also the possibility that velocity fluctuations induced by convected turbulence will result in the generation of additional noise-producing inhomogeneities in the form of vorticity shed from trailing edges and projections and subsequently swept downstream. Vortex shedding will occur in regions of high shear where viscous effects are significant and in a manner which tends to diminish the large gradients in the flow velocity which might otherwise occur. In treating such questions analytically the precise details of the vorticity generation process can often be ignored, for example the rate of production of vorticity at a trailing edge may be determined by requiring that the fluctuating velocity be finite at the edge (Kutta condition). This is the procedure used in classical thin-airfoil theory (see, for example, von Kármán & Sears 1938; Filotas 1969; Graham 1970; Mugridge 1971) in determining the fluctuating lift produced by a turbulent gust.

When the characteristic frequency of the disturbed motion is in the audible range there is considerable uncertainty regarding the validity of the Kutta condition. It would clearly be incorrect to impose a full Kutta condition at frequencies exceeding the characteristic inverse response time associated with viscous effects at an edge. However, Jones (1972) has examined a model problem involving the generation of sound by a stationary line source located in the vicinity of the trailing edge of a large airfoil, and reports no significant acoustic response arising from the imposition of the Kutta condition. This may be contrasted with the prediction of Crighton (1972) concerning the edge diffraction radiation induced by the unstable oscillations of a vortex wake. Crighton developed his analysis from earlier work of Orszag & Crow (1970), and concluded that, at low mean-flow Mach numbers M, the application of a Kutta condition at the edge resulted in an increase in the acoustic intensity, the sound pressure level varying as M^2 rather than M^4 (the dependence in the absence of additional vortex shedding). A similar dependence on the mean-flow Mach number has been predicted by Davies (1975), who applied Crighton's argument to the case of a large airfoil in the presence of a uniform mean flow. Recently Crighton & Leppington (1974) and Morgan (1974) considered the excitation of a semi-infinite vortex sheet by an impulsive line source located in fluid at rest. The Kutta condition was imposed, but apparently had no appreciable influence on the intensity of the radiated sound.

It is a major contention of the present paper that the conflicting conclusions of the investigations cited above have arisen because of inadequacies in the mathematical modelling of the interaction of a real aerodynamic source with a trailing edge. That this can be an important issue has been demonstrated quite recently by Howe (1975*a*), Ffowcs Williams & Lovely (1975) and Crighton (1975), who showed that the simple point-dipole representation of the acoustic effect of turbulence close to a compact rigid body (Curle 1955) leads to an incorrect prediction of the Doppler amplification of the sound in the presence of a mean flow. Similarly Dowling (1976) has shown by reference to simple model problems involving real sources that sound radiated at 90° to the flight path of an aircraft can experience amplification due to forward flight, contrary to earlier views based on the consideration of distributions of ideal acoustic dipoles and quadrupoles.

In this paper we examine a sequence of mathematical problems intended to model as realistically as possible the mechanism by which sound is generated as a turbulent eddy is convected in a mean flow past an airfoil or a sharp edge. In order to facilitate the analysis it will be assumed that the characteristic mean-flow Mach number M is sufficiently small that $M^2 \ll 1$. This permits the mean flow to be regarded as incompressible, with constant sound speed, but does not eliminate the convective effect of the flow on the generated sound, which is already present at $O(M)$. It also enables one to determine the flow in the neighbourhood of a compact airfoil and near the trailing edge of a large airfoil by means of the powerful techniques of classical, incompressible potential flow theory. The analysis is restricted to two-dimensional problems, the turbulence being modelled, in the main, by a convected *line vortex*, which may be regarded as a prototype two-dimensional turbulent eddy. An elegant discussion of problems of a similar nature has been given by Levine (1975), although the effects of a mean flow and vortex shedding were not included. Here, however, we shall be concerned principally with determining the manner in which the shedding of vorticity during the passage of the turbulent eddy past the trailing edge modifies the properties of the radiated sound.

In § 2 the general aerodynamic noise problem is formulated in terms of the Lighthill (1952) acoustic-analogy theory in the form developed by Howe (1975 *b*), which describes the generation of sound by vorticity and by entropy gradients. The two basic model problems to be considered are then described, and the equations are specialized to a form appropriate for their solution by means of the theory of Green's functions. That theory is then applied (§ 3) to problems of sound generation by turbulence convected past a compact airfoil, comparison being made with earlier work going back to von Kármán & Sears (1938). In § 4 the case of a non-compact airfoil modelled by a semi-infinite plate is examined when the mean flow is the same on both sides of the plate. Then (§ 5) the analysis is extended to cover the more interesting situation, relevant in any discussion of the problem of excess jet noise, in which the flow on either side of the plate has a different mean velocity, giving rise to the presence of a mean vortex wake. This characterizes the problem of sound generation by turbulence exhausting from the jet pipe of an aircraft engine in flight.

It is concluded that at low mean-flow Mach numbers the application of a Kutta condition, with the consequent smoothing of the disturbed flow in the vicinity of a trailing edge, always leads to a beneficial reduction in the level of the radiated sound.

2. Formulation of the aerodynamic sound problems

We consider an ideal gas and suppose that it is permissible to neglect dissipation due to viscosity and heat conduction during the passage of a turbulent eddy past the airfoil. The specific entropy S of a fluid particle therefore satisfies

$$DS/Dt = 0, \tag{2.1}$$

D/Dt being the material derivative with respect to time. The stagnation enthalpy is taken as the fundamental acoustic variable and is defined by

$$B = w + \tfrac{1}{2}v^2, \tag{2.2}$$

where w is the specific enthalpy and $\mathbf{v}$ the fluid velocity. B is determined in terms of the vorticity $\boldsymbol{\omega}$ and the entropy gradient ∇S by the inhomogeneous wave equation

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right) + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\,.\,\nabla - \nabla^2\right\}B = \operatorname{div}\{\boldsymbol{\omega}\wedge\mathbf{v} - T\nabla S\} - \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\,.\,(\boldsymbol{\omega}\wedge\mathbf{v} - T\nabla S) \tag{2.3}$$

(Howe 1975 *b*). Here c and T denote respectively the local speed of sound and the fluid temperature.

In regions of the flow exterior to those occupied by distributions of vorticity and entropy gradients, $p \equiv p(\rho)$, where ρ is the density, and the specific enthalpy becomes

$$w = \int \frac{dp}{\rho}. \tag{2.4}$$

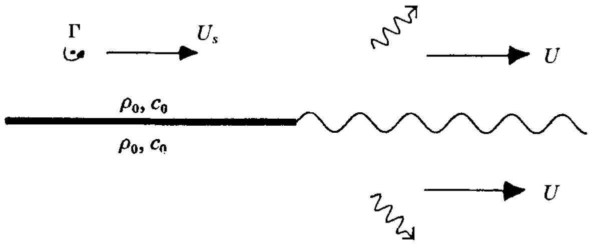

FIGURE 1. Generation of sound by a convected line vortex and shed vorticity. The airfoil is two-dimensional, of chord $2a$ and is set at zero incidence to the mean stream.

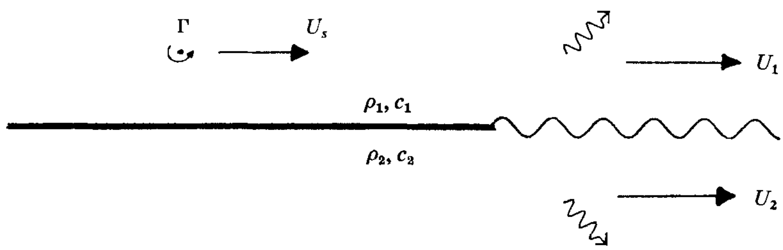

FIGURE 2. Generation of sound by a convected line vortex and wake vorticity in the case of a semi-infinite plate. The mean-flow velocity in the $+x_1$ direction is U_1 in $x_2 > 0$ and U_2 in $x_2 < 0$, with corresponding mean densities and sound speeds ρ_1, ρ_2 and c_1, c_2 respectively. The origin of co-ordinates is taken at the edge of the plate.

It follows that in a reference frame convecting at the local mean velocity the perturbation in B is p/ρ_0, where p is the acoustic pressure fluctuation and ρ_0 the local mean density.

Figures 1 and 2 illustrate the general characteristics of the problems to be examined. The classical problem of the gust loading of an airfoil located at rest in a stream of mean velocity U in the $+x_1$ direction is depicted in figure 1. The airfoil is assumed to be a rigid flat plate at zero incidence extending from $x_1 = -a$ to $x_1 = +a$ in the plane $x_2 = 0$. In all cases the model is taken to be two-dimensional, conditions being uniform in the x_3 direction, which is taken out of the paper. The incident gust or turbulent eddy consists of a line vortex of strength Γ parallel to the edge of the airfoil, although in § 3 the case of a spatially unbounded, harmonic gust is also examined. Detailed calculation will be undertaken only on the basis of linearized airfoil theory, which requires the vortex strength Γ to be sufficiently small that the path of the vortex is insensibly different from $x_2 = h =$ constant, during its passage past the leading and trailing edges of the airfoil. When a Kutta condition is imposed at the trailing edge there will be an additional distribution of shed vorticity in the wake, which is assumed to constitute an infinitely thin vortex sheet whose elements are swept downstream at the mean velocity U. The linearization also implies that the incident vortex convects at the mean stream velocity, although it is convenient to denote this convection velocity by U_s on the understanding that ultimately we must set $U_s = U$.

In figure 2 the finite airfoil is replaced by a semi-infinite rigid plate occupying

$x_1 < 0$, $x_2 = 0$. In the general case the mean flow velocities and mean fluid densities are respectively U_1, U_2 and ρ_1, ρ_2 on the upper ($x_2 > 0$) and lower ($x_2 < 0$) sides of the plate and the vortex sheet in the wake. Again, the two-dimensional linear theory of the excitation of the system by a line vortex Γ convected at velocity U_s past the edge of the plate along the path $x_2 = h > 0$ is considered. This problem models the generation of sound by turbulence swept out of an aircraft engine's jet pipe in flight. The particular case in which $U_1 = U_2 = U$ and $\rho_1 = \rho_2 = \rho_0$ is also examined, and provides a description of the generation of sound when the chord $2a$ of the airfoil of figure 1 greatly exceeds the characteristic acoustic wavelength. The frequency of the generated sound is of order U/h, corresponding to a wavelength $O(h/M)$, so that this case will be appropriate provided that $M \gg h/2a$.

In order to formulate these problems mathematically we must first determine the linearized forms of the inhomogeneous source terms on the right of (2.3). The incident line vortex Γ is specified by the singular distribution of vorticity

$$\boldsymbol{\omega} = \Gamma \mathbf{1} \delta(x_1 - U_s t)\, \delta(x_2 - h), \tag{2.5}$$

where $\mathbf{1}$ is a unit vector parallel to the x_3 axis. Observe that this particular choice for the incident eddy enables the case of an arbitrary two-dimensional distribution $\Omega(x_1 - U_s t,\, x_2)$ of frozen vorticity to be analysed by making use of the identity

$$\Omega(x_1 - U_s t, x_2) = \int \Omega(y_1, y_2)\, \delta(x_1 - y_1 - U_s t)\, \delta(x_2 - y_2)\, dy_1\, dy_2 \tag{2.6}$$

and the principle of superposition.

Thus, noting that the mean convection velocity U_s is steady, the linearized form of the source term associated with the incident vortex is precisely

$$\operatorname{div}(\boldsymbol{\omega} \wedge \mathbf{v}) \simeq \Gamma U_s\, \partial\{\delta(x_1 - U_s t)\, \delta(x_2 - h)\}/\partial x_2. \tag{2.7}$$

Since the unsteady motion induced by the passage of the vortex past the trailing edge can result in the shedding of additional vorticity and in the perturbation of an existing mean vortex wake, there is also a distribution of unsteady vorticity and entropy gradient, which on linear theory lies in the plane $x_2 = 0$, $x_1 > 0$. We assume that this wake may be adequately approximated by a vortex sheet, in which case $\boldsymbol{\omega} \wedge \mathbf{v} - T\nabla S$ vanishes on either side of the sheet, and may therefore be represented by a singular distribution

$$\boldsymbol{\omega} \wedge \mathbf{v} - T\nabla S = Z(\mathbf{x}, t)\, \delta\{x_2 - \eta(x_1, t)\}\, \mathbf{n} + \mathbf{w} \wedge \mathbf{v}_n, \quad \text{say}, \tag{2.8}$$

where $\mathbf{n}$ is the unit normal to the sheet, $\eta(x_1, t)$ its displacement from the mean position and $\mathbf{v}_n$ the normal velocity of the sheet (cf. the analogous discussion of entropy spots given by Howe 1975*b*). The strength Z of the singularity is obtained by integrating the momentum equation written in Crocco's form,

$$\partial \mathbf{v}/\partial t + \nabla B = -(\boldsymbol{\omega} \wedge \mathbf{v} - T\nabla S), \tag{2.9}$$

across a small interval normal to and enclosing the sheet. If $[\;]_2^1$ denotes the jump in crossing the sheet in the $+x_2$ direction, this gives

$$Z = \int (\boldsymbol{\omega} \wedge \mathbf{v} - T\nabla S) . d\mathbf{x} = -[B]_2^1, \tag{2.10}$$

since $\partial \mathbf{v}/\partial t$ is finite. But the flow on either side of the sheet is irrotational with potentials ϕ_1 and ϕ_2, say. Further, Bernoulli's equation implies that on either side

$$B \equiv \int \frac{dp}{\rho} + \tfrac{1}{2}v^2 = -\frac{\partial \phi}{\partial t}, \tag{2.11}$$

so that we may also write

$$Z = \frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t} \equiv \left[\frac{\partial \phi}{\partial t}\right]_2^1. \tag{2.12}$$

Now Z is generally non-zero even in the absence of an incident disturbance because of the presence of the mean vortex sheet and density gradient. Set

$$Z = -B_0 + [\partial \phi/\partial t]_2^1, \tag{2.13}$$

where the constant B_0 denotes the jump in the mean value of the stagnation enthalpy across the sheet, and the term in square brackets represents the perturbation produced by the eddy. Thus, using (2.7) and (2.8) in the right-hand side of the inhomogeneous wave equation (2.3), and linearizing with respect to the perturbation source terms, we obtain

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right) + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}.\nabla - \nabla^2\right\} B = \Gamma U_s \frac{\partial}{\partial x_2}\{\delta(x_1 - U_s t)\,\delta(x_2 - h)\} + \frac{\partial}{\partial x_2}\left\{\left[\frac{\partial \phi}{\partial t}\right]_2^1 \delta(x_2)\right\}$$
$$+ \frac{\partial}{\partial x_1}\{(U_1 - U_2)\,v_n\,\delta(x_2)\} - \operatorname{div}\{\nabla[B_0 H(f)]\} + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}.\nabla[B_0 H(f)], \tag{2.14}$$

the second and third terms on the right-hand side being non-zero only in the wake of the airfoil. In this result $f \equiv f(\mathbf{x}, t) \equiv x_2 - \eta(x_1, t)$ is the equation of the vortex sheet and $H(x)$ the Heaviside unit function. The sheet always lies along an instantaneous streamline, so that $Df/Dt = 0$ and the terms involving B_0 on the right of (2.14) may therefore be absorbed into the wave operator on the left-hand side. Hence, if B is re-defined as the perturbation in the stagnation enthalpy about its local mean value, we have simply

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right) + \frac{1}{c^2}\frac{D\mathbf{v}}{Dt}.\nabla - \nabla^2\right\} B = \Gamma U_s \frac{\partial}{\partial x_2}\{\delta(x_1 - U_s t)\,\delta(x_2 - h)\}$$
$$+ \frac{\partial}{\partial x_2}\left\{\left[\frac{\partial \phi}{\partial t}\right]_2^1 \delta(x_2)\right\} + \frac{\partial}{\partial x_1}\{(U_1 - U_2)\,v_n\,\delta(x_2)\}. \tag{2.15}$$

It is apparent from this that only fluctuations in the values of the wake source terms in a reference frame moving with the vortex sheet actually generate sound. This is a particular example of a general result obtained by Ffowcs Williams (1974).

At low mean-flow Mach numbers, c^2 may be assumed to be constant on each side of the vortex sheet, and when the wave operator is also linearized (2.15) reduces to

$$\left\{\frac{1}{c^2}\frac{D^2}{Dt^2} - \nabla^2\right\} B = \Gamma U_s \frac{\partial}{\partial x_2}\{\delta(x_1 - U_s t)\,\delta(x_2 - h)\} + \frac{\partial}{\partial x_2}\left\{\left[\frac{\partial \phi}{\partial t}\right]_2^1 \delta(x_2)\right\}$$
$$+ \frac{\partial}{\partial x_1}\{(U_1 - U_2)\,v_n\,\delta(x_2)\}, \tag{2.16}$$

where $D/Dt = \partial/\partial t + U\,\partial/\partial x_1$, U being the local mean-flow velocity.

In the absence of a discontinuity in the entropy S across the vortex sheet, it is convenient to have available an alternative representation of the wake source term of (2.16). The mean density is now uniform, and continuity of pressure implies that

$$[\partial\phi/\partial t]_2^1 = -[U\,\partial\phi/\partial x_1]_2^1 = U_2 u_2 - U_1 u_1. \tag{2.17}$$

In particular, when $U_2 = U_1 = U$,

$$[\partial\phi/\partial t]_2^1 = U(u_2 - u_1) = U\gamma(x_1, t), \tag{2.18}$$

where $\gamma(x_1, t)$ is the circulation density in the wake. Equation (2.16) now becomes

$$\left\{\frac{1}{c^2}\frac{D^2}{Dt^2} - \nabla^2\right\} B = \Gamma U_s \frac{\partial}{\partial x_2}\{\delta(x_2 - U_s t)\,\delta(x_2 - h)\} + U\frac{\partial}{\partial x_2}\{\gamma(x_1, t)\,\delta(x_2)\}. \tag{2.19}$$

To complete the mathematical formulation we now specify the conditions under which the radiated sound is to be determined. The acoustic response of a turbulent eddy in the presence of a scattering body is greatest when it is located well within a characteristic wavelength of the body (Crighton & Leppington 1971). In the problems under consideration that wavelength is $O(h/M)$, and will be assumed to be large compared with the chord $2a$ of the airfoil of figure 1, so enabling the flow in the neighbourhood of the airfoil which determines the fluctuating wake circulation $\gamma(x_1, t)$, to be regarded as incompressible. In the case of the semi-infinite plate of figure 2 we shall suppose that M is sufficiently small that the distance of the incident vortex from the edge of the plate is always much less than the acoustic wavelength. The characteristic wavelength is now of order R_0/M, where R_0 is the instantaneous distance of the vortex from the edge of the plate (cf. Howe 1975*b*, § 5), and this implies that the restriction will be true for all vortex positions if it is satisfied for $R_0 = h$. Again the properties of the vorticity may then be determined by means of incompressible flow theory.

After the incident vortex has passed the trailing edge and the shed vorticity has been swept downstream by the mean flow, the residual acoustic radiation arises solely from the nonlinear self-interaction of the vortex field. In two-dimensional, low Mach number situations the intensity of the sound is then proportional to M^7. In the neighbourhood of a compact airfoil (figure 1) or of the edge of a semi-infinite plate (figure 2) the intensity of the sound generated by a two-dimensional eddy generally varies respectively as M^5 and M^4 (Ffowcs Williams 1969), and we shall be interested in calculating only these more powerful components of the radiated sound.

In these circumstances the radiation may be calculated by making use of the appropriate *low frequency* Green's function (Howe 1975*a*). This method has been applied to an extensive range of aerodynamic noise problems by Howe (1975 *a*, *b*) and Ffowcs Williams & Howe (1975). The Green's function $G(\mathbf{x}, \mathbf{y}, t, \tau)$ is defined as the particular solution of

$$\left\{\frac{1}{c^2}\frac{D^2}{Dt^2} - \frac{\partial^2}{\partial x_j^2}\right\} G = \delta(\mathbf{x} - \mathbf{y})\,\delta(t - \tau) \tag{2.20}$$

which satisfies the radiation condition and the condition of vanishing normal derivative on the rigid surfaces of the airfoil. The solution $B(\mathbf{x}, t)$ of (2.20) in the

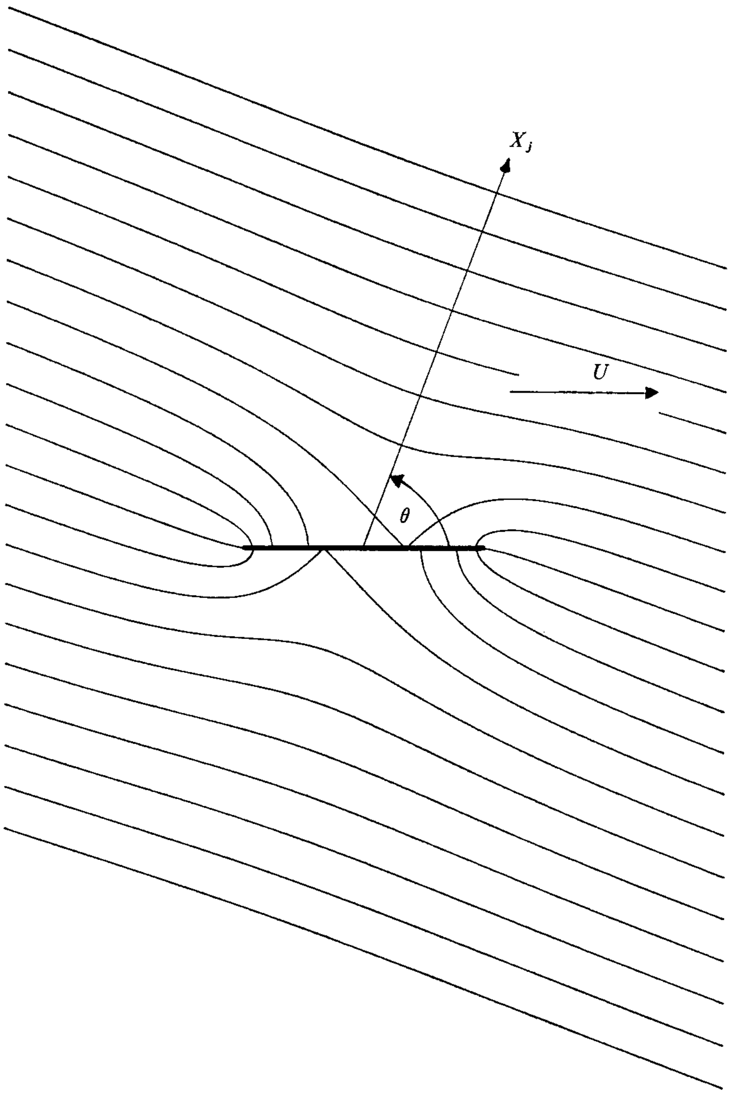

FIGURE 3. Illustrating the definition of the low frequency Green's function (2.22) for the compact airfoil. The airfoil lies between $x_1 = \pm a$ on the x_1 axis, and the figure shows the locus of points of constant X_j defined in (2.23) in the case in which the observer's direction $\theta = 70°$. At large distances $X_j \sim x_j$ and the loci become straight lines normal to the θ direction.

case in which the impulsive point source is replaced by a distribution $f(\mathbf{x}, t)$ is given by means of the convolution integral

$$B(\mathbf{x}, t) = \int f(\mathbf{y}, \tau)\, G(\mathbf{x}, \mathbf{y}, t, \tau)\, d^3\mathbf{y}\, d\tau. \tag{2.21}$$

For the airfoil of figure 1

$$G(\mathbf{x}, \mathbf{y}, t, \tau) \simeq \frac{1}{4\pi|\mathbf{X}-\mathbf{Y}|}\,\delta\left\{t-\tau-\frac{|\mathbf{X}-\mathbf{Y}|}{c}+\mathbf{M}.\frac{(\mathbf{X}-\mathbf{Y})}{c}\right\}, \tag{2.22}$$

where $\mathbf{M} = \mathbf{U}/c$, $X_3 = x_3$ and when j denotes a direction normal to the x_3 axis, X_j represents the potential of incompressible flow about the airfoil which at large distances is of unit velocity in the j direction (Howe 1975*a*). In particular, when this direction makes an angle θ with the direction of the mean flow (figure 3) and $z = x_1 + ix_2$, then

$$X_j = \mathrm{Re}\{z\cos\theta - i(z^2-a^2)^{\frac{1}{2}}\sin\theta\}, \tag{2.23}$$

so that $X_j \to \mathrm{Re}\{z\,e^{-i\theta}\} \equiv x_j$ as $x_j/a \to \infty$. The approximation involved in (2.22) is appropriate provided that $M^2 \ll 1$, the airfoil is compact, and either the source point $\mathbf{y}$ or the observation point $\mathbf{x}$ is located many characteristic wavelengths from the airfoil.

The corresponding form of the Green's function for the semi-infinite rigid plate of figure 2 is (Howe 1975*b*)

$$G(\mathbf{x},\mathbf{y},t,\tau) \simeq \frac{\phi^*(\mathbf{x})\,\phi^*(\mathbf{y})}{\pi|\mathbf{x}|}\delta\left\{t-\tau-\frac{|\mathbf{x}|}{c[1+\mathbf{M}.\mathbf{x}/|\mathbf{x}|]}\right\}. \tag{2.24}$$

This is only suitable for dealing with two-dimensional problems in which the observation point $\mathbf{x} = (x_1, x_2)$ is located many characteristic wavelengths from the edge of the plate and the source position $\mathbf{y}$ is well within a wavelength of the edge. The function $\phi^*(\mathbf{x})$ is the potential describing incompressible flow about a rigid half-plane:

$$\phi^*(\mathbf{x}) = R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta, \tag{2.25}$$

where $(x_1, x_2) = R(\cos\theta, \sin\theta)$. The argument of the δ-function in (2.24) depends on the mean-flow Mach number M. In the presence of a mean vortex sheet, as in figure 2, the appropriate value for M is that of the mean stream in which the observer is located. The Green's function (2.24) takes account of diffraction of the near field of the aerodynamic sources by the semi-infinite plate, but not of the subsequent scattering by the wake. That is a lower-order effect (see § 5) except for observation directions making grazing angles with the wake. But this is of no real significance, since in any event exact linear theory cannot account for the secular multiple scattering of the sound which occurs in the wake (see, for example, Howe 1974). It follows from (2.25) that $\partial\phi^*/\partial x_1$ vanishes identically in the wake $x_2 = 0$, $x_1 > 0$ of the semi-infinite plate; this implies that the final source term on the right of (2.16) makes no contribution to the radiated sound and so will be omitted.

3. Convection of turbulence past a compact airfoil

In the classical theory (von Kármán & Sears 1938) the fluctuating lift on a thin airfoil located at rest in a uniform mean stream of velocity U is calculated for a convected two-dimensional gust or eddy whose incompressible velocity perturbation is specified by

$$\mathbf{u} = (u, v) = \mathbf{A}\exp[i\{k_1(x_1-Ut)+k_2x_2\}], \tag{3.1}$$

$\mathbf{A}$ being a constant vector. In principle the case of an arbitrary two-dimensional eddy may then be analysed by superposition. This is also the method that is usually applied to determine the sound generated during the passage of an eddy past an airfoil (a general review is given by Goldstein 1974). Indeed, according to Curle's (1955) extension of Lighthill's theory, the fluctuating lift force on a compact body furnishes the strength of an equivalent acoustic dipole. This is different from the procedure outlined in the previous section, in which the sound is regarded as generated by the fluctuating vorticity, so that before proceeding to the discussion of the problem depicted in figure 1 we shall first establish the identity of our approach with the Curle theory.

Consider a line vortex of strength Γ' located in the z plane at the point with complex position $z_0 = x_1 + ix_2$. By means of the conformal transformation

$$2z = \zeta + a^2/\zeta \tag{3.2}$$

the airfoil in figure 1 is mapped into a circle of radius a in the ζ plane, and the line vortex into the point ζ_0, say. The complex potential in the ζ plane due to this vortex may be written in the form

$$w_{\Gamma'} = \frac{-i\Gamma'}{2\pi}\left\{\ln[\zeta - \zeta_0] - \ln\left[\zeta - \frac{a^2}{\zeta_0^*}\right] + \ln\zeta\right\}, \tag{3.3}$$

(von Kármán & Sears 1938), an expression which means that net circulation about the plate is absent. However the transformation (3.2) implies that the velocities at the edges of the plate are infinite. If a Kutta condition is to be imposed to remove the singularity at the trailing edge, vorticity must be generated there and shed into the wake, where it is swept downstream by the mean flow. Let $\bar{\gamma} = \gamma(\xi)\,d\xi$ denote the strength of an elementary shed vortex located at $z = \xi > 0$ on the x_1 axis. This lies at the point ξ', say, in the ζ plane and the appropriate form for the corresponding induced potential flow is

$$w_{\bar{\gamma}} = (-i\bar{\gamma}/2\pi)\{\ln[\zeta - \xi'] - \ln[\zeta - a^2/\xi']\}, \tag{3.4}$$

which satisfies the additional constraint imposed by Kelvin's theorem (Lamb 1932, p. 36) that the total circulation about the plate and the shed vortex should vanish. The complete expression for the velocity potential involves a summation over all of the shed vorticity:

$$w = w_{\Gamma'} + \Sigma w_{\gamma}. \tag{3.5}$$

The velocity will remain finite at the trailing edge $\zeta = z = a$ provided that $(dw/dz)_{z=a}$ is finite, and the above expressions imply that this will be the case if $\gamma(\xi)$ satisfies the von Kármán–Sears equation

$$\Gamma'\left\{\frac{1}{\zeta_0 - a} + \frac{1}{\zeta_0^* - a}\right\} + \int_a^\infty \frac{\gamma(\xi)}{a}\left(\frac{\xi + a}{\xi - a}\right)^{\frac{1}{2}} d\xi = 0, \tag{3.6}$$

where the integration is taken along the wake and

$$\zeta_0 = x_1 + ix_2 + [(x_1 + ix_2)^2 - a^2]^{\frac{1}{2}}. \tag{3.7}$$

To apply this result to the harmonic gust (3·1), note first of all that the incident vorticity

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{v} = i\mathbf{k}\wedge\mathbf{A}\exp[i\{k_1(x_1 - Ut) + k_2x_2\}], \tag{3.8}$$

where $\mathbf{k} = (k_1, k_2)$. Since this may also be expressed in the form

$$\boldsymbol{\omega} = i\mathbf{k}\wedge\mathbf{A}\int\exp[i\{k_1(X_1 - Ut) + k_2X_2\}]\,\delta(X_1 - x_1)\,\delta(X_2 - x_2)\,dX_1\,dX_2, \tag{3.9}$$

the disturbance induced by the vorticity distribution (3.8) is equivalent to that which would be produced by an infinite array of line vortices located at points $\mathbf{X} = (X_1, X_2)$. The strength of these vortices is proportional to $\mathbf{k} \wedge \mathbf{A}$, a vector of length $k_1 A_2 - k_2 A_1$ directed out of the paper. The gust is incompressible, which requires that $\mathbf{k}.\mathbf{A} \equiv k_1 A_1 + k_2 A_2$ should vanish, and it therefore follows that each elementary vortex is of strength

$$\Gamma'(\mathbf{X}) = iA_2(k_1^2+k_2^2)\,k_1^{-1}\exp\left[i\{k_1(X_1-Ut)+k_2X_2\}\right]dX_1dX_2. \tag{3.10}$$

The total shed vorticity must have the same harmonic time dependence and is convected at the mean-flow velocity. Hence we can set

$$\gamma \equiv \gamma(\xi, t) = \gamma_0 \exp\left[-ik_1 U\{t-\xi/U\}\right]. \tag{3.11}$$

Substituting this and (3.10) into the integral equation (3.6) we find that

$$\gamma_0 = \frac{-i\dfrac{A_2(k_1^2+k_2^2)}{k_1}\displaystyle\int_{-\infty}^{\infty}\left\{\frac{1}{\zeta-a}+\frac{1}{\zeta^*-a}\right\}e^{i\mathbf{k}.\mathbf{x}}\,d^2\mathbf{x}}{\displaystyle\int_a^{\infty}\exp\left(ik_1\xi\right)\left(\frac{\xi+a}{\xi-a}\right)^{\frac{1}{2}}\frac{d\xi}{a}}, \tag{3.12}$$

where ζ is defined by (3.7). The integrals in this result may be reduced by elementary means to standard representations of Bessel functions given in Abramowitz & Stegun (1965, p. 360), formally divergent integrals being interpreted where necessary as generalized functions. It is thereby deduced that

$$\gamma_0 = -4iA_2\{J_0(k_1a)+iJ_1(k_1a)\}/\{H_0^{(1)}(k_1a)+iH_1^{(1)}(k_1a)\}. \tag{3.13}$$

We now have an expression defining the strength of the wake vorticity source term $\gamma(x_1, t)$ on the right of the inhomogeneous wave equation (2.19). Substituting for this term from (3.11) and using the low frequency Green's function (2.22) for a compact airfoil, the contribution B_w made by the wake to the acoustic radiation is given by the convolution integral

$$B_w = \frac{U\gamma_0}{4\pi}\iiint_{-\infty}^{\infty} dy_2\,dy_3\,d\tau\int_a^{\infty} dy_1\frac{\partial}{\partial y_2}\left(\exp\{-ik_1U(\tau-y_1/U)\}\,\delta(y_2)\right) \\ \times\frac{1}{|\mathbf{X}-\mathbf{Y}|}\,\delta\left\{t-\tau-\frac{|\mathbf{X}-\mathbf{Y}|}{c}+\frac{\mathbf{M}.(\mathbf{X}-\mathbf{Y})}{c}\right\}. \tag{3.14}$$

Taking account of the definition (2.23) of Y_j, we have for observation points $\mathbf{x}$ in the distant field

$$B_w \simeq \frac{-U\gamma_0}{4\pi c}\frac{\partial}{\partial t}\int_a^{\infty}dy_1\int_{-\infty}^{\infty}dy_3 \\ \times\frac{y_1\sin\theta\,\exp\left[i\{k_1y_1-k_1U(t-|\mathbf{x}-y_3\mathbf{1}|/c+\mathbf{M}.\mathbf{x}/c)\}\right]}{(y_1^2-a^2)^{\frac{1}{2}}\,|\mathbf{x}-y_3\mathbf{1}|}, \tag{3.15}$$

where θ is the angle between the observer's direction and the $+x_1$ axis. The y_3 integral is performed by the method of stationary phase, and the integral with respect to y_1 reduces to

$$\int_a^{\infty}\frac{y_1}{(y_1^2-a_2)^{\frac{1}{2}}}\exp\left(ik_1y_1\right)dy_1 = -i\frac{\partial}{\partial k_1}\int_1^{\infty}\frac{\exp\left(ik_1a\lambda\right)}{(\lambda^2-1)^{\frac{1}{2}}}\,d\lambda = -\frac{\pi a}{2}H_1^{(1)}(k_1a) \tag{3.16}$$

(Abramowitz & Stegun 1965, p. 360). It follows that

$$B_w \simeq \frac{-iU\gamma_0 a \sin\theta}{4}\left(\frac{\pi k_1 M}{2R}\right)^{\frac{1}{2}} H_1^{(1)}(k_1 a)\exp\left[-i\{k_1 U[t]-\tfrac{1}{4}\pi\}\right], \tag{3.17}$$

where here and henceforth terms in square brackets are to be evaluated at the retarded time $t-R/c(1+M\cos\theta)$, R being the distance of the observer from the airfoil.

Similarly the vorticity distribution (3.8) determines the contribution made by the incident vorticity to the radiated sound. Inserting (3.8) into the vortical source terms on the right of the general equation (2.3), we have in the linear approximation

$$\operatorname{div}(\boldsymbol{\omega}\wedge\mathbf{v}) = \frac{iA_2 U(k_1^2+k_2^2)}{k_1}\frac{\partial}{\partial x_2}(\exp\left[i\{k_1(x_1-Ut)+k_2 x_2\}\right]), \tag{3.18}$$

which is distributed throughout the whole of space. The convolution integral is now performed as before for an observer in the distant field, the final integration with respect to y_1 reducing to a Bessel function, and we find that B_Γ, the radiation from the incident vorticity, is given by

$$B_\Gamma \simeq A_2 U a \sin\theta\left(\frac{\pi k_1 M}{2R}\right)^{\frac{1}{2}} J_1(k_1 a)\exp\left[-i\{k_1 U[t]-\tfrac{1}{4}\pi\}\right]. \tag{3.19}$$

Equations (3.17) and (3.19) give the separate contributions to the radiation field arising respectively from the wake and the incident turbulent gust. The corresponding acoustic pressure fluctuation p is obtained by noting that Bernoulli's equation (2.11) (with $\frac{1}{2}v^2 = U\,\partial\phi/\partial x$) implies that correct to the neglect of terms of order M^2 relative to unity

$$B = (p/\rho_0)(1+M\cos\theta). \tag{3.20}$$

Thus, adding (3.17) and (3.19) and making use of (3.13) and a well-known Wronskian for Bessel functions (Abramowitz & Stegun 1965, p. 360), it follows that

$$\frac{p}{\rho_0} \simeq \frac{iA_2 U\sin\theta}{(1+M\cos\theta)}\left(\frac{2M}{\pi k_1 R}\right)^{\frac{1}{2}}\frac{\exp\left[-i\{k_1 U[t]-\frac{1}{4}\pi\}\right]}{\{H_0^{(1)}(k_1 a)+iH_1^{(1)}(k_1 a)\}}. \tag{3.21}$$

This result coincides with that obtained by use of the fluctuating lift force to determine the strength of an equivalent acoustic dipole (the details are discussed by Goldstein 1974), and establishes the identity between the direct method presented above and that due to Curle (1955).

It is a simple matter to extend the analysis to the problem of figure 1. The incident vortical eddy is now the line vortex of (2.5). When a Kutta condition is imposed at the trailing edge the strength of the shed vorticity may be obtained by first expressing (2.5), with $U_s = U$, in the form of a linear array of harmonic line vortices:

$$\boldsymbol{\omega} = \frac{\Gamma}{2\pi U}\mathbf{1}\iint_{-\infty}^{\infty}\delta(x_1-y_1)\,\delta(x_2-h)\exp\{i\omega(y_1/U-t)\}\,dy_1\,d\omega. \tag{3.22}$$

Let $\gamma(\xi,\omega)\,e^{-i\omega t}$ be the component of the wake vorticity of frequency ω. Then use of (3.22) in the von Kármán–Sears equation (3.6) shows that $\gamma(\xi,\omega)$ is determined by

$$\frac{\Gamma}{2\pi U}\int_{-\infty}^{\infty}\left\{\frac{1}{\zeta-a}+\frac{1}{\zeta^*-a}\right\}\exp\left(i\frac{\omega y_1}{U}\right)dy_1+\int_a^\infty\frac{\gamma(\xi,\omega)}{a}\left(\frac{\xi+a}{\xi-a}\right)^{\frac{1}{2}}d\xi = 0, \quad (3.23)$$

where $\zeta = y_1+ih+\{(y_1+ih)^2-a^2\}^{\frac{1}{2}}$. Setting, as before,

$$\gamma(\xi,\omega) = \gamma_0(\omega)\exp\{i\omega\xi/U\}, \quad (3.24)$$

it follows by methods entirely analogous to those discussed above that

$$\gamma_0(\omega) = \frac{-\Gamma\,\mathrm{sgn}\,(\omega)}{\pi U}\exp\left(\frac{-|\omega|\,h}{U}\right)\left(\frac{J_0(\omega a/U)+iJ_1(\omega a/U)}{H_0^{(1)}(\omega a/U)+iH_1^{(1)}(\omega a/U)}\right). \quad (3.25)$$

For each value of ω the acoustic field due to the wake vorticity is given by the convolution integral (3.14) with γ_0 replaced by $\gamma_0(\omega)$.

The total sound field generated by the wake then follows by integration with respect to ω. Making the substitution $\mu = \omega h/U$ and using (3.20), one finds for the acoustic pressure perturbation

$$\frac{p}{\rho_0}\simeq\frac{i\Gamma Ua\sin\theta\, e^{\frac{1}{4}i\pi}}{4\pi h(1+M\cos\theta)}\left(\frac{M\pi}{2Rh}\right)^{\frac{1}{2}}$$
$$\times\int_{-\infty}^{\infty}\mu^{\frac{1}{2}}\,\mathrm{sgn}\,(\mu)\exp\left(-|\mu|-i\mu U\,[t]\right)/hH_1^{(1)}\left(\frac{\mu a}{h}\right)\left(\frac{J_0(\mu a/h)+iJ_1(\mu a/h)}{H_0^{(1)}(\mu a/h)+iH_1^{(1)}(\mu a/h)}\right)d\mu, \quad (3.26)$$

where $\mu^{\frac{1}{2}} = +i\,|\mu|^{\frac{1}{2}}$ for $\mu < 0$.

This integral may be evaluated explicitly in the two extreme cases (i) $a/h \ll 1$ and (ii) $a/h \gg 1$, corresponding respectively to a vortex passing at large and small distances from the airfoil. To do this we use the appropriate asymptotic approximations to the Bessel functions in (3.26), and obtain in case (i)

$$\frac{p}{\rho_0}\simeq\frac{\Gamma U\sin\theta}{4(1+M\cos\theta)}\left(\frac{M}{2Ra}\right)^{\frac{1}{2}}\left[\left(\frac{a}{R_0}\right)^{\frac{3}{2}}\sin\left(\frac{3\theta_0}{2}\right)\right], \quad (3.27)$$

where (R_0, θ_0) are the polar co-ordinates of the incident vortex, θ_0 being measured from the $+x_1$ direction, and the square brackets imply evaluation at the retarded time $t-R/c(1+M\cos\theta)$.

In case (ii), $a/h \gg 1$, (3.26) reduces to

$$\frac{p}{\rho_0}\simeq\frac{\Gamma U\sin\theta}{4\pi(1+M\cos\theta)}\left(\frac{M}{Ra}\right)^{\frac{1}{2}}\left[\frac{a/h}{\{1+(Ut-a)^2/h^2\}}\right], \quad (3.28)$$

where $x_1 = U[t]$, $x_2 = h$ is the retarded vortex position.

Turn attention now to the direct radiation produced by the line vortex convecting past the airfoil, and corresponding to the first term on the right of (2.19) with the convection velocity $U_s = U$. Using the representation

$$\delta(x_1-Ut) = \frac{1}{2\pi U}\iint_{-\infty}^{\infty}\delta(x_1-y_1)\exp\left\{i\omega\left(\frac{y_1}{U}-t\right)\right\}dy_1\,d\omega$$

as in (3.22), it follows by convolution in the manner already described that in the distant field the pressure fluctuations associated with the vortex are given by

$$\frac{p}{\rho_0} \simeq \frac{-iU\Gamma a \sin\theta}{4\pi h(1+M\cos\theta)}\left(\frac{M\pi}{2Rh}\right)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} \times \int_{-\infty}^{\infty} \mu^{\frac{1}{2}} \operatorname{sgn}(\mu) \exp\left(-|\mu| - i\frac{U\mu}{h}[t]\right) J_1\left(\frac{\mu a}{h}\right) d\mu. \quad (3.29)$$

In the first of the asymptotic cases considered above ($a/h \ll 1$), we have

$$\frac{p}{\rho_0} \simeq \frac{-3\Gamma U \sin\theta}{16(1+M\cos\theta)}\left(\frac{M}{2Ra}\right)^{\frac{1}{2}}\left[\left(\frac{a}{R_0}\right)^{\frac{5}{2}} \sin\left(\frac{5\theta_0}{2}\right)\right], \quad (3.30)$$

which is seen to be an order of magnitude a/R_0 smaller than the corresponding wake contribution (3.27). Of more interest, however, is the field strength in the case $a/h \gg 1$, when the vortex passes close to the airfoil. We now find

$$\frac{p}{\rho_0} \simeq \frac{\Gamma U \sin\theta}{4\pi(1+M\cos\theta)}\left(\frac{M}{Ra}\right)^{\frac{1}{2}}\left[\frac{(Ut+a)\,a/h^2}{\{1+(Ut+a)^2/h^2\}} - \frac{a/h}{\{1+(Ut-a)^2/h^2\}}\right]. \quad (3.31)$$

The two contributions in the retarded-time brackets of this result may be interpreted as accounting respectively for the diffractive effects of the leading and trailing edges of the airfoil. Observe, however, that the dependence of the radiation is not symmetric in this respect, because in two-dimensional problems of the present type the instantaneous intensity of the field is determined by a weighted average over the entire previous history of the source distribution (Ffowcs Williams 1969). The significant feature of (3.31) is that the relatively strong diffraction field from the trailing edge $x_1 = U[t] = a$ is exactly cancelled by the acoustic field (3.28) produced by the wake vorticity when the Kutta condition is imposed. This is a first indication of the importance of vortex shedding in influencing the intensity of the sound generated by convected turbulence.

In conclusion we note that all of the radiation fields predicted in this section have $p \sim \rho_0 u U M^{\frac{1}{2}}$, u and U being respectively the perturbation and mean-flow velocities, which is characteristic of the two-dimensional, linear aerodynamic noise fields generated by turbulence near a compact solid.

4. Convection of turbulence past a semi-infinite plate in a uniform mean flow

The analysis of the previous section may be extended in an obvious manner to cover the problem of figure 2 when the mean-flow velocities U_1 and U_2 are both equal to U and $\rho_1 = \rho_2 = \rho_0$. Let a line vortex of strength Γ' lie in the z plane at $z = z_0 = X_1 + ih$, the origin being located at the edge of the plate. The flow is mapped into the upper half of the ζ plane by means of the transformation $\zeta = iz^{\frac{1}{2}}$, so that the complex potential of the flow induced by the vortex Γ' and a shed vorticity distribution $\gamma(\xi)$ in the wake of the plate is readily seen to be given by

$$w = \frac{-i\Gamma'}{2\pi}\ln\left\{\frac{z^{\frac{1}{2}} - z_0^{\frac{1}{2}}}{z^{\frac{1}{2}} + z_0^{*\frac{1}{2}}}\right\} - \frac{i}{2\pi}\int_0^{\infty} \gamma(\xi) \ln\left\{\frac{z^{\frac{1}{2}} - \xi^{\frac{1}{2}}}{z^{\frac{1}{2}} + \xi^{\frac{1}{2}}}\right\} d\xi. \quad (4.1)$$

Imposing the Kutta condition at the edge of the plate we find that $\gamma(\xi)$ is determined by the equation

$$\int_0^\infty \frac{\gamma(\xi)}{\xi^{\frac{1}{2}}}\,d\xi + \frac{\Gamma'}{2}\left\{\frac{1}{(X_1+ih)^{\frac{1}{2}}} + \frac{1}{(X_1-ih)^{\frac{1}{2}}}\right\} = 0. \tag{4.2}$$

Introduce the representation (3.22) of the incident line vortex (with U set equal to the convection velocity U_s); then for each frequency component ω,

$$\gamma(\xi) \equiv \gamma(\xi,\omega)\,e^{-i\omega t} = \gamma_0(\omega)\exp\{i\omega(\xi/U - t)\}, \tag{4.3}$$

where

$$\gamma_0(\omega)\int_0^\infty \frac{\exp(i\omega\xi/U)\,d\xi}{\xi^{\frac{1}{2}}} + \frac{\Gamma}{4\pi U_s}\int_{-\infty}^{\infty}\left\{\frac{1}{(X_1+ih)^{\frac{1}{2}}} + \frac{1}{(X_1-ih)^{\frac{1}{2}}}\right\}\exp\left(i\frac{\omega X_1}{U_s}\right)dX_1 = 0. \tag{4.4}$$

Performing the integration with respect to ξ we have

$$\gamma_0(\omega) = \frac{-\Gamma\omega^{\frac{1}{2}}\,e^{-\frac{1}{4}i\pi}}{4\pi U_s(U\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left\{\frac{1}{(X_1+ih)^{\frac{1}{2}}} + \frac{1}{(X_1-ih)^{\frac{1}{2}}}\right\}\exp\left(i\frac{\omega X_1}{U_s}\right)dX_1. \tag{4.5}$$

The acoustic radiation produced by the wake for each ω is now obtained by substituting (4.3) into the wake source term of (2.19), and convoluting the resulting expression with the low frequency half-plane Green's function (2.24). Thus the distant pressure perturbations are given by the integral

$$\frac{p(\omega)}{\rho_0} \simeq \frac{U\phi^*(\mathbf{x})}{\pi|\mathbf{x}|(1+M\cos\theta)}\int\frac{\partial}{\partial y_2}\left\{\delta(y_2)H(y_1)\gamma_0(\omega)\exp\left(i\omega\left(\frac{y_1}{U}-\tau\right)\right)\right\}$$
$$\times\,\phi^*(\mathbf{y})\,\delta\left\{t-\tau-\frac{|\mathbf{x}|}{c(1+M\cos\theta)}\right\}d^2\mathbf{y}\,d\tau, \tag{4.6}$$

where use has been made of (3.20) and where $M = U/c$. Observing that $\partial\phi^*(\mathbf{y})/\partial y_2 = \frac{1}{2}/y_1^{\frac{1}{2}}$ on $y_2 = 0$, $y_1 > 0$, we find

$$\frac{p(\omega)}{\rho_0} \simeq \frac{-U\gamma_0(\omega)\sin\frac{1}{2}\theta}{2(1+M\cos\theta)}\left(\frac{U}{\pi R\omega}\right)^{\frac{1}{2}}\exp\{-i(\omega[t]-\tfrac{1}{4}\pi)\}. \tag{4.7}$$

Replacing $\gamma_0(\omega)$ by (4·5) and integrating with respect to ω we have for the wake radiation

$$\frac{p}{\rho_0} \simeq \frac{\Gamma U\sin\frac{1}{2}\theta}{4\pi R^{\frac{1}{2}}(1+M\cos\theta)}\int_{-\infty}^{\infty}\left\{\frac{1}{(X_1+ih)^{\frac{1}{2}}} + \frac{1}{(X_1-ih)^{\frac{1}{2}}}\right\}$$
$$\times\,\delta\left(X_1 - U_s\left\{t-\frac{R}{c(1+M\cos\theta)}\right\}\right)dX_1$$
$$= \frac{\Gamma U\sin\frac{1}{2}\theta}{2\pi R^{\frac{1}{2}}(1+M\cos\theta)}\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right], \tag{4.8}$$

where the term in square brackets is evaluated at the retarded position (R_0, θ_0) of the incident vortex.

The direct radiation from the incident vortex is obtained by convoluting the first source term on the right of the inhomogeneous wave equation (2.19) with the half-plane Green's function, and this gives in a straightforward manner

$$\begin{aligned}\frac{p}{\rho_0} &= \frac{\Gamma U_s \phi^*(\mathbf{x})}{\pi|\mathbf{x}|(1+M\cos\theta)}\int\frac{\partial}{\partial y_2}\{\delta(y_2-h)\,\delta(y_1-U_s\tau)\}\\ &\quad\times\phi^*(\mathbf{y})\,\delta\left\{t-\tau-\frac{|\mathbf{x}|}{c(1+M\cos\theta)}\right\}d^2\mathbf{y}\,d\tau\\ &= \frac{-\Gamma U_s\sin\frac{1}{2}\theta}{\pi R^{\frac{1}{2}}(1+M\cos\theta)}\left[\frac{\partial\phi^*}{\partial x_2}\right].\end{aligned} \tag{4.9}$$

the gradient being evaluated at the retarded position of the vortex. An elementary calculation shows that

$$\partial\phi^*/\partial x_2 = \cos(\tfrac{1}{2}\theta_0)/2R_0^{\frac{1}{2}},$$

so that (4.9) becomes

$$\frac{p}{\rho_0} \simeq \frac{-\Gamma U_s\sin\frac{1}{2}\theta}{2\pi R^{\frac{1}{2}}(1+M\cos\theta)}\left[\frac{\cos\frac{1}{2}\theta}{R_0^{\frac{1}{2}}}\right]. \tag{4.10}$$

Comparing this result with the wake radiation (4.8) we see that in the present approximation the acoustic field *vanishes* identically when the convection velocity of the incident vortex assumes its actual value $U_s = U$. To place this result in context recall that terms nonlinear in the perturbation velocities have been neglected. Each contribution to the radiation (4.8) and (4.10) is of order $p/\rho_0 \sim uU$, where u is a fluctuation velocity, and our result therefore indicates that in order to obtain radiation fields with the same parametric dependence on the flow velocity it is necessary to take explicit account of the departure of the path of the incident vortex from the rectilinear one $x_2 = h$. This is an essentially nonlinear effect produced by image vortices in the plate and by the shed vorticity. In other words, when the Kutta condition is imposed the relatively strong radiation (4.10) due to the mean-flow convection of the eddy past the trailing edge is annihilated.

This conclusion is considerably more dramatic than the partial cancellation already noted in relation to the compact airfoil. In both cases the effect of vortex shedding is to eliminate the strong diffraction radiation from the edge where the shedding occurs. Let us examine in more detail the nature of the local incompressible flow which accompanies this phenomenon. The imposition of the Kutta condition reduces the x_2 component of the perturbation velocity to zero at the trailing edge. When the incident vortex translates past the semi-infinite plate at the *velocity of the mean stream*, the application of the Kutta condition also implies the vanishing of the x_2 component of the velocity at *all points of the vortex sheet which constitutes the wake*. It is perhaps of interest to present a detailed demonstration of this remarkable result.

To do this use the first term in the velocity potential (4.1) to show that the x_2 component of the wake velocity induced at a distance $x_1 > 0$ from the edge of the plate by the incident vortex is given by

$$v_s(x_1) = \frac{\Gamma R_0^{\frac{1}{2}}(x_1 - R_0)\cos\frac{1}{2}\theta_0}{2\pi x_1^{\frac{1}{2}}\{x_1^2 - 2x_1 R_0 \cos\theta_0 + R_0^2\}}. \tag{4.11}$$

The corresponding velocity component v due to the wake vorticity is determined by the second term on the right of (4.1), and reduces to the principal-value integral

$$v(x_1) = \frac{1}{2\pi x_1^{\frac{1}{2}}} \int_0^\infty \frac{\gamma(\xi)\,\xi^{\frac{1}{2}}\,d\xi}{x_1 - \xi}. \tag{4.12}$$

The wake vorticity $\gamma(\xi)$ is obtained by integrating (4.3) with respect to ω. Thus, taking account of the definition (4.5) of $\gamma_0(\omega)$, we have

$$v(x_1) = \frac{-\Gamma\, e^{-\frac{1}{4}i\pi}}{8\pi^2 U_s(\pi x_1 U)^{\frac{1}{2}}} \int_0^\infty d\xi \iint_{-\infty}^{\infty} \left\{\frac{1}{(y_1 + ih)^{\frac{1}{2}}} + \frac{1}{(y_1 - ih)^{\frac{1}{2}}}\right\} \times \frac{(\omega\xi)^{\frac{1}{2}}}{(x_1 - \xi)} \exp\left\{i\omega\left(\frac{y_1}{U_s} + \frac{\xi}{U} - t\right)\right\} dy_1\, d\omega. \tag{4.13}$$

The integration with respect to y_1 may be performed by means of the substitution $\mu = (y_1 \pm ih)^{\frac{1}{2}}$, and yields

$$\int_{-\infty}^{\infty} \left\{\frac{1}{(y_1 + ih)^{\frac{1}{2}}} + \frac{1}{(y_1 - ih)^{\frac{1}{2}}}\right\} \exp\left(i\frac{\omega y_1}{U_s}\right) dy_1 = 2\left(\frac{\pi U_s}{\omega}\right)^{\frac{1}{2}} \exp\left(i\frac{\pi}{4} - \frac{|\omega| h}{U_s}\right). \tag{4.14}$$

The ω integral is then straightforward and reduces (4.13) to

$$v(x_1) = \frac{i\Gamma}{4\pi^2}\left(\frac{U}{U_s x_1}\right)^{\frac{1}{2}} \int_0^\infty \frac{\xi^{\frac{1}{2}}}{(x_1 - \xi)} \left\{\frac{1}{\xi - z_0 U/U_s} - \frac{1}{\xi - z_0^* U/U_s}\right\} d\xi, \tag{4.15}$$

where $z_0 = U_s t + ih$ is the position of the vortex at time t. This may be evaluated by recalling that the principal value is equal to the mean of the two integrals passing just above and just below the pole at $\xi = x_1 > 0$. The path of integration of the first of these is rotated in an anticlockwise sense through 360° about the origin, the resulting change in sign of $\xi^{\frac{1}{2}}$ thereby annulling the second integral. The only contribution to (4.15) then arises from the poles of the integrand encountered during rotation, and we obtain

$$v(x_1) = \frac{-\Gamma R_0^{\frac{1}{2}}(U/U_s)\,(x_1 - UR_0/U_s)\cos\frac{1}{2}\theta_0}{2\pi x_1^{\frac{1}{2}}\{x_1^2 - 2(Ux_1 R_0/U_s)\cos\theta_0 + (U/U_s)^2 R_0^2\}}. \tag{4.16}$$

It is now clear that this velocity is equal and opposite to that induced by the incident vortex and given in (4.11) only when the translation velocity of the vortex U_s equals the mean stream velocity U. Thus, in the linear approximation (with $U_s = U$), the vortex convects past the edge of the plate where just sufficient vorticity is shed to ensure that the normal velocity vanishes at all points of the plane $x_2 = 0$. The flow conditions are then steady in a frame at rest relative to the vortex, so that no sound is radiated.

We end this section with a discussion of the relation of the above analysis to previous work (Jones 1972) on the interaction of aerodynamic noise sources with a semi-infinite plate in the presence of a uniform mean flow. In particular we consider very briefly the situation in which the incident vortex dipole source is replaced by a *stationary* harmonic line monopole of strength m, represented by a singularity of the form $m\delta(x_1-X_1)\,\delta(x_2-X_2)\exp(-i\omega t)$ on the right of the inhomogeneous wave equation (2.19).

Allowing for the possibility of vortex shedding at the edge of the plate, a procedure analogous to that leading from (4.1) to (4.2) yields the following integral equation for the distribution $\gamma(\xi)$ of circulation in the wake:

$$\int_0^\infty \frac{\gamma(\xi)}{\xi^{\frac{1}{2}}}\,d\xi = \frac{m}{R_0^{\frac{1}{2}}}\sin\left(\frac{\theta_0}{2}\right)e^{-i\omega t}, \tag{4.17}$$

where (R_0, θ_0) are the polar co-ordinates of the line source. The substitution (4.3) gives

$$\gamma_0 = \frac{m}{R_0^{\frac{1}{2}}}\sin\left(\frac{\theta_0}{2}\right)\left(\frac{\omega}{\pi U}\right)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}, \tag{4.18}$$

and when this is inserted into (4.7) we find that in the distant field the pressure fluctuation induced by the wake is given by

$$\frac{p}{\rho_0} \simeq \frac{-mU\sin\frac{1}{2}\theta\sin\frac{1}{2}\theta_0}{2\pi(1+M\cos\theta)\,(RR_0)^{\frac{1}{2}}}\exp\left\{-i\omega\left(t-\frac{R}{c(1+M\cos\theta)}\right)\right\}. \tag{4.19}$$

Correct to an error of order M relative to unity, this agrees with the result obtained by Jones (1972), who applied the Wiener–Hopf technique to the equations of linear compressible flow theory.

5. Interaction of convected turbulence with a semi-infinite mean vortex sheet

We now turn to a consideration of the general problem of figure 2 in which the vortex is located in a mean stream of velocity U_1 and density ρ_1 and convects past the edge of the semi-infinite plate at velocity U_s. The flow in the 'ambient' medium $x_2 < 0$ has mean velocity U_2 and density ρ_2. The local details of the flow in the neighbourhood of the edge of the plate are determined by means of linearized incompressible flow theory, as before, and this requires that the mean-flow Mach numbers satisfy $M_1^2, M_2^2 \ll 1$.

The analysis of this problem is complicated by the inherent Kelvin–Helmholtz instability of the mean vortex sheet. Accordingly, before proceeding to the general case we first examine the nature of this difficulty by reference to a simplified canonical problem discussed by Crighton (1972).

Sound generation by a vortex-sheet instability

A vortex sheet leaving a semi-infinite plate executes a two-dimensional, Kelvin–Helmholtz, spatially growing instability motion. In terms of the notation of figure 2, it is assumed that $\rho_1 = \rho_2 = \rho_0$, say, $U_1 = 0$, and that the unstable motion is harmonic in time with radian frequency $\omega > 0$. There is no incident disturbing vortex or eddy. Such a mode of oscillation constitutes an eigensolution of the problem.

In the particular case in which the Kutta condition is not imposed at the edge of the plate, Crighton obtains an integral expression for linearized compressible potential flow on either side of the vortex sheet, and deduces that, in the limit of small mean-flow Mach number M_2, the acoustic pressure perturbation far from the edge and in the region $x_2 > 0$ where there is no mean flow is given by

$$\frac{p}{\rho_0} = -A\left(\frac{\omega U_2 2^{\frac{1}{2}}}{\pi R}\right)^{\frac{1}{2}} . \sin\left(\frac{\theta}{2}\right)\exp\left\{-i\omega\left(t-\frac{R}{c}\right)+\frac{i\pi}{8}\right\}. \tag{5.1}$$

In this expression (R, θ) are the polar co-ordinates of the distant observation point, as defined in the previous section, and an error in the sign of Crighton's original result has been corrected (Crighton 1976, private communication). The constant A denotes the amplitude of the exponentially growing instability mode [see equation (5.3) below]. A spatially growing mode of this form is possible even in the absence of a plate, but in that case the disturbed flow on either side decays exponentially with distance from the mean position of the sheet. The sound field (5.1) is therefore produced through the interaction of the instability wave with the edge of the plate. Indeed the detailed calculations of Möhring (1975) and of Bechert & Michel (1975) reveal that this interaction is restricted essentially to points of the flow lying within a distance of order U_2/ω from the edge of the plate.

Define κ and $\bar{\kappa}$ by

$$\kappa = \frac{\omega}{U_2}(1+i), \quad \bar{\kappa} = \frac{\omega}{U_2}(1-i). \tag{5.2}$$

When the sound speed is allowed to become infinite in Crighton's (1972) exact solution, we obtain representations of the potentials ϕ_1 and ϕ_2 which respectively describe incompressible flow in $x_2 \gtrless 0$. On the mean position ($x_2 = 0$, $x_1 > 0$) of the vortex sheet these assume the following simplified forms:

$$\left.\begin{aligned} \phi_1 &= A\exp\{i(\bar{\kappa}x_1-\omega t)\} - A\exp\{i(\kappa x_1 - \omega t - \tfrac{1}{4}\pi)\}, \\ \phi_2 &= -iA\exp\{i(\bar{\kappa}x_1-\omega t)\} - iA\exp\{i(\kappa x_1 - \omega t - \tfrac{1}{4}\pi)\}, \end{aligned}\right\} \tag{5.3}$$

the first term on the right of each equation growing exponentially with distance downstream.

According to the inhomogeneous wave equation (2.16) (in which Γ is set equal to zero and the term in $\mathbf{v}_n$ is discarded), the discontinuity specified by (5.3) in the perturbation potential across the vortex sheet results in the generation of sound. In the previous section the sound was determined by convoluting the wake source with the low frequency half-plane Green's function (2.24). A difficulty arises in the present case in applying such a procedure because the exponential growths of ϕ_1 and ϕ_2 with distance from the edge of the plate imply the divergence of the convolution integral.

Now it has already been noted that, in the absence of the plate, the subsonic, linearized compressible equations of motion admit exact time-harmonic solutions growing exponentially with distance x_1 along the sheet, but decaying exponentially with distance on either side of the sheet. A disturbance of this form does not correspond to a sound wave. On the other hand, when the motion of the boundary $x_2 = 0$ produced by the instability is taken as specified, the problem of determining the flow on either side of the sheet may then be posed. The solution can be expressed as a convolution integral of the free-space Green's function and the boundary motion (see, for example, Morse & Ingard 1968, p. 366), and again we encounter the difficulty of interpreting a formally divergent integral. In this case the problem is resolved by temporarily removing the exponential growth by allowing the frequency ω to assume a complex value, the result for real ω being subsequently obtained by analytic continuation.

This procedure, which is discussed in more detail below, may be applied to Crighton's problem by requiring that ω should lie in the wedge $\frac{1}{4}\pi < \arg\omega < \frac{3}{4}\pi$. The second source term on the right of (2.16) becomes for $x_1 > 0$

$$\frac{\partial}{\partial x_2}\left\{\left[\frac{\partial\phi}{\partial t}\right]_2^1 \delta(x_2)\right\} = -i\omega A\frac{\partial}{\partial x_2}\{([1+i]\exp\{i(\bar{\kappa}x_1-\omega t)\} - [1-i]\exp\{i(\kappa x_1-\omega t-\tfrac{1}{4}\pi)\})\,\delta(x_2)\}. \tag{5.4}$$

Convoluting this with the low frequency half-plane Green's function (2.24) (with $M_1 \equiv 0$ for an observer in $x_2 > 0$), we obtain in the distant field

$$\begin{aligned}\frac{p}{\rho_0} &\simeq -\frac{i\omega A\phi^*(\mathbf{x})}{\pi|\mathbf{x}|}\int_0^\infty dy_1\iint_{-\infty}^\infty \frac{\partial}{\partial y_2}\big(\delta(y_2)\,([1+i]\exp\{i(\bar{\kappa}y_1-\omega\tau)\} \\ &\qquad -[1-i]\exp\{i(\bar{\kappa}y_1-\omega\tau-\tfrac{1}{4}\pi)\})\big)\,\phi^*(\mathbf{y})\,\delta\left\{t-\tau-\frac{|\mathbf{x}|}{c}\right\}dy_2\,d\tau \\ &= \frac{i\omega A\sin\frac{1}{2}\theta}{2\pi R^{\frac{1}{2}}}\exp\left\{-i\omega\left(t-\frac{R}{c}\right)\right\}\int_0^\infty\big((1+i)\exp(i\bar{\kappa}y_1) \\ &\qquad -(1-i)\exp\{i(\kappa y_1-\tfrac{1}{4}\pi)\}\big)\frac{dy_1}{y_1^{\frac{1}{2}}}.\end{aligned} \tag{5.5}$$

Making the substitution $\mu = y_1^{\frac{1}{2}}$ and using (5.2), the integral becomes

$$2i(U_2\pi 2^{\frac{1}{2}}/\omega)^{\frac{1}{2}}\,e^{\frac{1}{8}i\pi},$$

which on substitution into (5.5) reduces the result to Crighton's low Mach number approximation (5.1).

Note that, as mentioned in § 2, the use of the Green's function (2.24) implicitly neglects the scattering of the sound generated near the edge of the plate by the vortex sheet. The agreement of our result with (5.1) indicates that the approximation involved is adequate, and would be expected to involve an error which is $O(M)$ smaller.

Linear theory of the incompressible motion of a line vortex near a semi-infinite vortex sheet

We now return to the general problem of figure 2. It is required to determine the sound radiated into the ambient flow region $x_2 < 0$ when the line vortex convects past the edge of the plate. The first step involves the determination of the incompressible flow in the neighbourhood of the edge. Related problems have been investigated by Orszag & Crow (1970), and more recently by Möhring (1975) and Bechert & Michel (1975). Möhring's analysis entailed a rather general, function theoretic discussion, and is not easily modified to deal with the present case. The work of Bechert & Michel, on the other hand, did not take explicit account of the infinite set of eigensolutions of the semi-infinite vortex sheet. Actually Crighton & Leppington (1974) and Morgan (1974) have examined the compressible problem in which an impulsive line source is located in fluid at rest. Unfortunately their results contain *ultradistributions*, which, together with the absence of a mean flow, make them difficult to interpret in terms of a more realistic aerodynamic source problem.

Let (u_1, v_1) and (u_2, v_2) denote the $(1,2)$ components of the perturbation velocities at points respectively on the upper and lower surfaces of the plate and the vortex sheet. These velocities are related by the vorticity equation

$$\operatorname{curl}\{\rho D\mathbf{v}/Dt\} = 0, \tag{5.6}$$

which in linearized form reduces to

$$\rho_1\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x_1}\right)u_1-\rho_2\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x_1}\right)u_2 = 0 \tag{5.7}$$

on the mean position $x_2 = 0$, $x_1 > 0$ of the vortex sheet. The kinematic condition that the sheet should coincide with an instantaneous streamline is

$$\left(\frac{\partial}{\partial t}+U_2\frac{\partial}{\partial x_1}\right)v_1 = \left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x_1}\right)v_2, \tag{5.8}$$

which is formally valid along the whole of the x_1 axis, since v_1 and v_2 vanish identically for $x_1 < 0$.

It will be assumed that all components of the perturbation velocities have at worst *integrable* singularities at the edge of the plate. The imposition of a Kutta condition will lead to the additional requirement that v_1 and v_2 vanish at the edge.

Next observe that the incompressible flow in $x_2 < 0$ contains no singularities, so that Cauchy's theorem applied to a large semicircular contour in the lower half of the $z = x_1+ix_2$ plane provides the following well-known relation between $u_2(x_1)$ and $v_2(x_1)$:

$$u_2(x_1) = \frac{1}{\pi}\int_0^\infty \frac{v_2(\xi)\,d\xi}{\xi-x_1}, \tag{5.9}$$

the principal value integral being taken along the $+x_1$ axis (cf. Bechert & Michel 1975). In order to obtain a similar connexion between the velocity components on the side $x_2 = +0$ of the x_1 axis, we first write (u_1, v_1) in the form

$$u_1 = u_0+u_s, \quad v_1 = v_0+v_s. \tag{5.10}$$

Here (u_s, v_s) is defined to be the velocity perturbation due to the incident vortex in the absence of a mean vortex sheet, i.e. the perturbation expressed by the first term of (4.1). Then u_s-iv_s, regarded as a function of the complex variable z, has a simple pole at the position $z_0 = U_s t+ih$ of the vortex, and a branch cut extending along the real axis from $-\infty$ to the edge of the plate. Thus the appropriate contour of integration for the application of Cauchy's theorem is the same as that leading to (5.9), and we have

$$_2u_s(x_1) = \frac{1}{\pi}\int_0^\infty \frac{v_s(\xi)\,d\xi}{\xi-x_1}, \tag{5.11}$$

where the notation ${}_2u_s(x_1)$ indicates that, for $x_1 < 0$, ${}_2u_s(x_1)$ is the velocity perturbation on the *lower* side $x_2 = -0$ of the plate. If ${}_1u_s(x_1)$ is the x_1 component of the vortex-induced velocity on $x_2 = +0$, then clearly for

$$x_1 > 0, \; {}_1u_s(x_1) = {}_2u_s(x_1).$$

The remaining component $u_0 - iv_0$ of the complex velocity on $x_2 = +0$ must represent the boundary value of a function which is regular in the upper half z plane, and in this case we have

$$u_0(x_1) = -\frac{1}{\pi}\int_0^\infty \frac{v_0(\xi)\,d\xi}{\xi - x_1}. \tag{5.12}$$

Introduce the definition

$$\mathscr{U}_1(x_1) = u_0(x_1) + {}_2u_s(x_1), \tag{5.13}$$

in which case $\mathscr{U}_1(x_1) \equiv u_1(x_1)$ for $x_1 > 0$. Since we have also

$$\mathscr{U}_1(x_1) = u_1(x_1) + \{{}_2u_s(x_1) - {}_1u_s(x_1)\}, \tag{5.14}$$

it follows that, if $u_1(x_1)$ is integrable at the edge of the plate, so is $\mathscr{U}_1(x_1)$ because ${}_2u_s(x_1) - {}_1u_s(x_1)$ possesses an inverse square-root singularity there (Lamb 1932, chap. 4). The definitions (5.10) and (5.13) may now be combined with (5.11) and (5.12) to give

$$\mathscr{U}_1(x_1) = -\frac{1}{\pi}\int_0^\infty \frac{v_1(\xi)\,d\xi}{\xi - x_1} + \frac{2}{\pi}\int_0^\infty \frac{v_s(\xi)\,d\xi}{\xi - x_1}. \tag{5.15}$$

Thus, since $u_1(x_1) \equiv \mathscr{U}_1(x_1)$ in $x_1 > 0$, (5.7)–(5.9) and (5.15) constitute a complete system sufficing to determine the motion of the vortex sheet under the driving influence of the line vortex expressed through the velocity component $v_s(x_1)$. It is assumed that $v_s(x_1)$ is prescribed, and that a causal solution is required.

The solution is time dependent because of the dependence of $v_s(x_1)$ on the instantaneous location of the vortex. In order to solve the problem we employ the double Fourier transform $f(k)$ of a function $f(x_1, t)$ defined by the reciprocal relations

$$\left.\begin{aligned} f(k) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} f(x_1, t)\exp\{-i(kx_1 - \omega t)\}\,dx_1\,dt, \\ f(x_1, t) &= \iint_{-\infty}^{\infty} f(k)\exp\{i(kx_1 - \omega t)\}\,dk\,d\omega, \end{aligned}\right\} \tag{5.16}$$

where, for convenience, the dependence of $f(k)$ on ω is not shown explicitly. It is also necessary to make use of the following notation familiar in applications of the Wiener–Hopf technique (Noble 1958):

$$f(k) = f^+(k) + f^-(k), \tag{5.17}$$

where

$$\left.\begin{aligned} f^+(k) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} f(x_1, t)\exp\{-i(kx_1 - \omega t)\}\,dt, \\ f^-(k) &= \frac{1}{(2\pi)^2}\int_{0}^{\infty} dx_1 \int_{-\infty}^{\infty} f(x_1, t)\exp\{-i(kx_1 - \omega t)\}\,dt, \end{aligned}\right\} \tag{5.18}$$

in which case $f^{\pm}(k)$ are respectively regular functions in the upper and lower halves of the k plane. It should be noted that these definitions involve the assumption that the integrals do in fact converge as $t \to \pm\infty$. We ensure that this is the case by assuming that the incident vortex is created instantaneously at time $t = t_0$, which is subsequently allowed to tend to $-\infty$, and by assuming further that ω lies in a suitable region of the upper half-plane in order for convergence to be achieved at $t = +\infty$. The satisfaction of the causality condition requires that the Fourier time transform be regular in the upper half ω plane (Lighthill 1960), so that, having solved the problem for this particular choice of ω, the solution is extended to the whole plane by analytic continuation. This is the procedure used by Crighton & Leppington (1974).

Take the Fourier transforms of (5.7) and (5.8):

$$\rho_1(\omega - U_1 k)\, u_1^-(k) - \rho_2(\omega - U_2 k)\, u_2^-(k) = C, \tag{5.19}$$

$$(\omega - U_2 k)\, v_1^-(k) - (\omega - U_1 k)\, v_2^-(k) = 0, \tag{5.20}$$

where $C \equiv C(\omega) = \rho_1 U_1 u_1(x_1, \omega) - \rho_2 U_2 u_2(x_1, \omega)$ evaluated at $x_1 = +0$, and is independent of k, since (5.7) requires that any integrable singularities in the terms of this difference are equal. Similarly, noting that $u_1^-(k) \equiv \mathscr{U}_1^-(k)$, the transforms of (5.9) and (5.15) are

$$u_2(k) \equiv u_2^+(k) + u_2^-(k) = i\,\mathrm{sgn}\,(k)\, v_2^-(k), \tag{5.21}$$

$$\mathscr{U}_1(k) \equiv \mathscr{U}_1^+(k) + u_1^-(k) = -i\,\mathrm{sgn}\,(k)\, v_1^-(k) + 2i\,\mathrm{sgn}\,(k)\, v_s^-(k). \tag{5.22}$$

In order to apply the usual Wiener–Hopf argument we introduce the identity

$$\mathrm{sgn}\, k = \lim_{\epsilon \to +0} (k^2 + \epsilon^2)^{\frac{1}{2}}/k, \tag{5.23}$$

where $(k^2+\epsilon^2)^{\frac{1}{2}} = (k+i\epsilon)^{\frac{1}{2}}(k-i\epsilon)^{\frac{1}{2}}$ and branch cuts are taken in the lower and upper half-planes respectively in such a manner that $(k \pm i\epsilon)^{\frac{1}{2}} \to +k^{\frac{1}{2}}$ as $k \to \infty$ along the positive real axis. The analysis is conducted for non-zero ϵ, after which the limit implied in (5.23) is taken.

Substituting for $u_1^-(k)$ and $u_2^-(k)$ from (5.21) and (5.22) into (5.19), we have for real k

$$\begin{aligned} -i(k-i\epsilon)^{-\frac{1}{2}}\{\rho_2(\omega - U_2 k)\, v_2^-(k) + \rho_1(\omega - U_1 k)\, v_1^-(k) - 2\rho_1(\omega - U_1 k)\, v_s^-(k)\} \\ = (k+i\epsilon)^{-\frac{1}{2}}\{C + \rho_1(\omega - U_1 k)\,\mathscr{U}_1^+(k) - \rho_2(\omega - U_2 k)\, u_2^+(k)\}. \end{aligned} \tag{5.24}$$

By construction the left-hand side of this equation is regular in the lower half of the k plane and the right-hand side is regular in the upper half-plane. Together they therefore define a function which is regular everywhere. Also the perturbation velocities possess only integrable singularities at the edge of the plate, and it follows (Noble 1958, p. 36) that their half-range transforms are at most finite as $k \to \infty$. Thus both sides of (5.24) grow no faster than $k^{\frac{1}{2}}$ at infinity. The usual Wiener–Hopf argument involving Liouville's theorem then implies that both sides are actually constant, and we may write

$$\rho_1(\omega - U_1 k)\, v_1^-(k) + \rho_2(\omega - U_2 k)\, v_2^-(k) = 2\rho_1(\omega - U_1 k)\, v_s^-(k) + P(k - i\epsilon)^{\frac{1}{2}}, \tag{5.25}$$

P being an arbitrary constant.

If $v_s(x_1)$ is set equal to zero, so that there is no external forcing of the vortex sheet, the solution of our equations for non-zero P corresponds to the eigensolution discussed by Crighton (1972) and at the beginning of this section. Equations (5.21) and (5.25) imply that in this case $v_1^-(k)$ and $v_2^-(k)$ behave like $k^{-\frac{1}{2}}$ for large k, a condition which indicates that $v_1(x_1)$ and $v_2(x_1)$ possess inverse square-root singularities at the edge of the plate (Noble 1958, p. 36). It therefore follows that it is impossible to impose a Kutta condition in the absence of external forcing. On the other hand, when the incident vortex is present it has already been noted that $v_s(x_1) = O(x_1^{-\frac{1}{2}})$ at the edge, which means that $v_s^-(k) \sim \alpha/(k-i\epsilon)^{\frac{1}{2}}$, say, for large k. Thus the Kutta condition may now be satisfied if P is chosen such that

$$P = 2\alpha\rho_1 U_1. \tag{5.26}$$

Combining (5.20) and (5.25) we have

$$\left.\begin{aligned} v_1^-(k) &= \frac{(\omega-U_1k)\{2\rho_1(\omega-U_1k)\,v_s^-(k)+P(k-i\epsilon)^{\frac{1}{2}}\}}{\{\rho_1(\omega-U_1k)^2+\rho_2(\omega-U_2k)^2\}},\\ v_2^-(k) &= \frac{(\omega-U_2k)\{2\rho_1(\omega-U_1k)\,v_s^-(k)+P(k-i\epsilon)^{\frac{1}{2}}\}}{\{\rho_1(\omega-U_1k)^2+\rho_2(\omega-U_2k)^2\}},\end{aligned}\right\} \tag{5.27}$$

and corresponding expressions for $\mathscr{U}_1(k)$ and $u_2(k)$ can be written down by making use of (5.21)–(5.23). Observe that the condition of regularity of $v_1^-(k)$ and $v_2^-(k)$ in the lower half of the k plane requires that ω lie in an appropriate region of the upper half-plane. Indeed, it may readily be shown that the zeros of the denominator in (5.27) lie at the conjugate points

$$k = \omega(\alpha \pm i\beta), \tag{5.28}$$

where

$$\alpha + i\beta = \frac{1+i(\rho_1/\rho_2)^{\frac{1}{2}}}{U_2+iU_1(\rho_1/\rho_2)^{\frac{1}{2}}}, \tag{5.29}$$

so that the corresponding singularities will be located in the upper half of the k plane if $\chi < \arg\omega < \pi-\chi$, where $\chi = \arg(\alpha+i|\beta|)$.

We are now in a position to write down a formal expression for the Fourier transform of $[\partial\phi/\partial t]_2^1$, the strength of the non-trivial wake source term on the right of the inhomogeneous wave equation (2.16). Denote this transform by $S'(k)$; then

$$S'(k) = (-\omega/k)\{u_1(k)-u_2(k)\}. \tag{5.30}$$

But (5.15) shows that in the wake ($x_1 > 0$) the inverse transforms of $u_1(k)$ and $\mathscr{U}_1(k)$ are the same function of x_1. Hence we may take

$$S(k) = (-\omega/k)\{\mathscr{U}_1(k)-u_2(k)\} \tag{5.31}$$

instead of $S'(k)$, a result which may also be expressed in the form

$$S(k) = i\omega k^{-2}(k^2+\epsilon^2)^{\frac{1}{2}}\{v_1^-(k)+v_2^-(k)-2v_s^-(k)\}. \tag{5.32}$$

Substituting for $v_1^-(k)$ and $v_2^-(k)$ from (5.27), it follows that for $x_1 > 0$, the Fourier time transform (denoted by a tilde) of $[\partial\phi/\partial t]_2^1$ is just

$$\begin{aligned}\widetilde{\left[\frac{\partial\phi}{\partial t}\right]_2^1} &= \int_{-\infty}^{\infty} S(k)\,e^{ikx}\,dk\\ &= \int_{-\infty}^{\infty}\frac{i\omega(k^2+\epsilon^2)^{\frac{1}{2}}}{k^2}\\ &\quad\times\left\{\frac{[2\rho_1(\omega-U_1k)\,v_s^-(k)+P(k-i\epsilon)^{\frac{1}{2}}]\,[2\omega-(U_1+U_2)\,k]}{[\rho_1(\omega-U_1k)^2+\rho_2(\omega-U_2k)^2]}-2v_s^-(k)\right\}e^{ikx}\,dk.\end{aligned} \tag{5.33}$$

We shall see below that for $x_1 > 0$ this integral can be evaluated by displacing the contour of integration in the k plane to $+i\infty$, the value of the integral being determined by simple poles of the integrand. Two of these poles correspond to the zeros of $\rho_1(\omega - U_1 k)^2 + \rho_2(\omega - U_2 k)^2$ given by (5.28), which for real ω characterize the Kelvin–Helmholtz instability of the vortex sheet. The final determination of $[\partial\phi/\partial t]_2^1$ then involves the calculation of inverse Fourier time transforms of the form

$$I = \int_{-\infty}^{\infty} F(\omega) \exp\left[i\omega\{(\alpha + i\beta)x_1 - t\}\right] d\omega, \tag{5.34}$$

where $F(\omega)$ is a known function of ω which is analytic in the upper half-plane. The range of integration is no longer restricted to the wedge within which ω was originally confined to ensure convergence of the Fourier integral (5.16), and it is apparent that in its present form (5.34) actually diverges.

This divergence of the integral representation of the solution is related to the difficulty already encountered at the beginning of this section. It has been recognized in a similar context by Jones & Morgan (1972, 1974) and by Crighton & Leppington (1974), who examined the case of an impulsively excited vortex sheet, and resolved the difficulty by the introduction of ultradistributions.

The reason for the divergence can be understood by first noting that $\rho_1(\omega - U_1 k)^2 + \rho_2(\omega - U_2 k)^2$ is the operational form of a partial differential operator. If $\eta(x_1, t)$ denotes the elevation of the vortex sheet above its mean position $x_2 = 0$, then after any initial disturbance η satisfies the equation

$$\rho_1\left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x_1}\right)^2 \eta + \rho_2\left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x_1}\right)^2 \eta = 0. \tag{5.35}$$

This is an elliptic partial differential equation with complex characteristics (see, for example, Garabedian 1964, p. 614) and the question of ascertaining the response $\eta(x_1, t)$ of the sheet to some initial disturbance constitutes what Hadamard (1952, § 21) has termed an incorrectly posed Cauchy problem. Such problems have arisen, for example, in connexion with the determination of the stand-off distance of a bow shock wave (Garabedian & Lieberstein 1958; Lieberstein 1959) and in the nonlinear theory of weakly modulated deep-water waves (Howe 1967). Their analysis generally proceeds by way of the method of characteristics; in the present case these are straight lines in the three-dimensional space spanned by time and the real and imaginary components of x_1. The existence of a solution thus depends on the possibility of analytically continuing the initial conditions of the problem to complex values of x_1. The fact that arbitrarily small changes in the definition of a function for real values of x_1 can produce exponentially large, if not singular behaviour in the complex plane is a reflexion of the basic instability of these problems.

In the problem of the semi-infinite vortex sheet considered here it is now clear that a meaningful solution is possible only if the initial data, specified in this instance by the vortex-induced velocity component $v_s(x_1)$, possess an appropriate analytic continuation. Thus the presence of β in the integral representation (5.34) implies that the solution of the problem is determined by the initial data, contained implicitly in $F(\omega)$, evaluated at complex values of the argument.

When a solution exists it is determined by calculating the integral in (5.34) for $\beta = 0$, the general case being obtained subsequently by analytic continuation. This is equivalent to introducing a continuation (or shift) operator

$$\exp\{-i\beta x_1 \partial/\partial t\}$$

and writing (5.34) in terms of the convergent integral

$$I = \exp\left\{-i\beta x_1 \frac{\partial}{\partial t}\right\} \int_{-\infty}^{\infty} F(\omega) \exp\{i\omega(\alpha x_1 - t)\} d\omega. \tag{5.36}$$

Generation of sound by a convecting line vortex

We are now ready to calculate the sound radiated when the vortex of figure 2 convects past the edge of the plate. This is given by the solution of the wave equation (2.16), with $[\partial\phi/\partial t]_2^1$ determined by the inverse Fourier time transform of (5.33). The first term on the right of (2.16) accounts for the direct radiation from the line vortex. The corresponding pressure perturbation in the ambient medium ($x_2 < 0$) is obtained by convolution with the half-plane Green's function (2.24) in the manner described in § 4, and we have [cf. (4.10)]

$$\frac{p_\Gamma}{\rho_2} = \frac{-\Gamma U_s \sin\frac{1}{2}\theta}{2\pi(1+M_2\cos\theta) R^{\frac{1}{2}}} \left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right], \tag{5.37}$$

(R, θ) being the polar co-ordinates of the observation point in $x_2 < 0$ and (R_0, θ_0) those of the line vortex at the retarded time $t - R/c_2(1+M_2\cos\theta)$.

In order to use (5.33) to determine the radiation from the wake we must first calculate the transform $v_s^-(k)$ of the velocity $v_s(x_1, t)$ induced by the line vortex on $x_2 = 0$, $x_1 > 0$ in the absence of a mean vortex sheet. To do this introduce a representation of the line vortex in terms of an infinite array of harmonic vortices distributed along $x_2 = h$, viz.

$$\begin{aligned} \boldsymbol{\omega} &= \frac{\Gamma \mathbf{1}}{2\pi U_s} \int_{-\infty}^{\infty} d\omega \int_{y_0}^{\infty} dy_1\, \delta(x_1 - y_1)\, \delta(x_2 - h) \exp\left\{i\omega\left(\frac{y_1}{U_s} - t\right)\right\} \\ &\equiv \Gamma \mathbf{1} H(t - y_0/U_s)\, \delta(x_1 - U_s t)\, \delta(x_2 - h), \end{aligned} \tag{5.38}$$

where $y_0 = U_s t_0$, t_0 being the time at which the incident vortex is created, which is subsequently allowed to tend to $-\infty$.

Using (5.38) and the known form for $v_s(x_1)$ for each of the harmonic line vortices [cf. (4.11)], it is a straightforward matter to show that

$$\begin{aligned} v_s^-(k) &= \frac{-i\Gamma(k - i\epsilon)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}}{16\pi^{\frac{3}{2}}(\omega - U_s k)} \int_{y_0}^{\infty} \left\{\frac{1}{(y_1+ih)^{\frac{1}{2}}} + \frac{1}{(y_1-ih)^{\frac{1}{2}}}\right\} \exp\left(i\frac{\omega y_1}{U_s}\right) dy_1 \\ &= \frac{-iU_s(k-i\epsilon)^{\frac{1}{2}}}{2(\omega - U_s k)} J(\omega), \quad \text{say}, \end{aligned} \tag{5.39}$$

where

$$J(\omega) = \frac{\Gamma e^{\frac{1}{4}i\pi}}{8\pi^{\frac{3}{2}} U_s} \int_{y_0}^{\infty} \left\{\frac{1}{(y_1+ih)^{\frac{1}{2}}} + \frac{1}{(y_1-ih)^{\frac{1}{2}}}\right\} \exp\left(i\frac{\omega y_1}{U_s}\right) dy_1. \tag{5.40}$$

Substituting for $v_s^-(k)$ in (5.33) we have

$$\widetilde{\left[\frac{\partial\phi}{\partial t}\right]_2^1} = \int_{-\infty}^{\infty} \frac{i\omega}{(k+i\epsilon)^{\frac{1}{2}}}\left\{\frac{[P-i\rho_1 U_s J(\omega)(\omega-U_1 k)/(\omega-U_s k)][2\omega-[(U_1+U_2)k]}{\rho_1(\omega-U_1k)^2+\rho_2(\omega-U_2k)^2} + \frac{iU_s J(\omega)}{(\omega-U_s k)}\right\}\exp(ikx_1)\,dk, \tag{5.41}$$

a result which demonstrates explicitly that the integrand is regular in the upper half-plane except for simple poles at $k = \omega/U_s$, $\omega(\alpha \pm i\beta)$ provided that ω is restricted to lie within the wedge defined by the inequality following (5.29).

Consider first the contribution to the radiated sound from the second term in the curly brackets in (5.41). This has a simple pole at $k = \omega/U_s$, but no contribution from the poles due to the Kelvin–Helmholtz instability. Evaluating the residue, the inverse Fourier time transform then gives the contribution ${}_2[\partial\phi/\partial t]_2^1$, say, of this term to the wake source strength $[\partial\phi/\partial t]_2^1$, viz.

$${}_2\left[\frac{\partial\phi}{\partial t}\right]_2^1 = 2\pi i\int_{-\infty}^{\infty}(\omega U_s)^{\frac{1}{2}}J(\omega)\exp\left\{i\omega\left(\frac{x_1}{U_s}-t\right)\right\}d\omega. \tag{5.42}$$

Substituting this into (2.16) and performing the convolution with the half-plane Green's function (2.24) we find that the corresponding contribution p_2 to the wake-generated sound is just

$$\frac{p_2}{\rho_2} = \frac{\Gamma\sin\frac{1}{2}\theta}{8\pi^2(1+M_2\cos\theta)R^{\frac{1}{2}}}\int_{y_0}^{\infty}dy_1\int_{-\infty}^{\infty}d\omega\left\{\frac{1}{(y_1+ih)^{\frac{1}{2}}}+\frac{1}{(y_1-ih)^{\frac{1}{2}}}\right\}\exp\left\{i\omega\left(\frac{y_1}{U_s}-[t]\right)\right\}, \tag{5.43}$$

where use has been made of (5.40) and $[t]$ denotes the retarded time

$$t - R/c_2(1+M_2\cos\theta).$$

The integration with respect to ω yields a δ-function, after which it follows that when $y_0 \to -\infty$

$$\frac{p_2}{\rho_2} = \frac{\Gamma U_s \sin\frac{1}{2}\theta}{2\pi(1+M_2\cos\theta)R^{\frac{1}{2}}}\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right], \tag{5.44}$$

the term in square brackets being evaluated at the retarded position of the line vortex.

Comparing this with the expression (5.37) for the direct vortex sound p_Γ, we see that the combination $p_\Gamma + p_2$ vanishes identically for arbitrary vortex convection velocity U_s, and therefore that the net radiated sound intensity is determined solely by the wake source strength ${}_1[\partial\phi/\partial t]_2^1$, corresponding to the first term in the curly brackets of (5.41).

Recall that we have not as yet imposed a Kutta condition at the edge of the plate. Reference to (5.27) and the steps leading to (5.41) reveals that the imposition of this condition implies that the constant P be chosen in such a way that

$$P - i\rho_1 U_s J(\omega)\left\{\frac{\omega-U_1k}{\omega-U_sk}\right\} \to 0$$

as $k \to \infty$, which means that

$$P = i\rho_1 U_1 J(\omega). \tag{5.45}$$

These conclusions clarify the significance of the vortex convection velocity U_s. When U_s equals the velocity U_1 of the mean stream in which the line vortex is embedded, it is apparent that the choice (5.45) of P involves also the vanishing of the first term in the curly brackets of (5.41) for all values of k. Consequently the corresponding contribution p_1 to the acoustic pressure field is identically zero. We have already seen that $p_\Gamma + p_2 = 0$, and we therefore arrive at the remarkable prediction that, on the present linearized theory, *when the Kutta condition is imposed, no sound is produced during the passage of the vortex past the edge of the plate.* This is a generalization of the result obtained in § 4 in the absence of a mean vortex wake, and indicates that any sound which is in fact produced arises as a result of nonlinear effects, and is then of order $\rho_0 v^2$, where $v \ll U$ is the characteristic perturbation velocity.

It is also clear that when $U_s = U_1$ the right-hand sides of equations (5.27) are also zero, thereby implying further that the x_2 components $v_1(x_1, t)$ and $v_2(x_1, t)$ of the perturbation velocities vanish at all points of the plate and wake. Again this is in accord with the result of § 4. Thus, in the linearized approximation, the imposition of the Kutta condition leads to a situation in which the perturbed motion of the fluid is steady in a frame convecting with the line vortex, in which case no sound is produced.

To complete the analysis of this section we now determine the radiation for $U_s \neq U_1$ in the two cases in which (i) a Kutta condition is *not* imposed, for which we set $P = 0$, and (ii) the condition is imposed by the choice of P given in (5.45). The details of the calculations are similar to those outlined above, formally divergent convolution integrals being evaluated by means of the interpretation (5.36), which is equivalent to the procedure used at the beginning of this section in the discussion of Crighton's (1972) problem of the semi-infinite vortex sheet.

Case (i). *No Kutta condition imposed.* The acoustic pressure at (R, θ) in $x_2 < 0$ is given by

$$\frac{p}{\rho_2} = \frac{-\Gamma U_s \sin\frac{1}{2}\theta}{\pi(1+M_2\cos\theta)R^{\frac{1}{2}}}\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right], \tag{5.46}$$

the term in square brackets being evaluated at the retarded position of the vortex. If the direct radiation from the vortex (5.37) is subtracted from this result, it follows that the additional sound produced by the presence of the wake is

$$\frac{p}{\rho_2} \simeq \left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)\frac{\Gamma U_s \sin\frac{1}{2}\theta}{2\pi(1+M_2\cos\theta)R^{\frac{1}{2}}}\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right]. \tag{5.47}$$

This term arises from the interaction between the incompressible edge flow induced by the line vortex and the infinite density gradient across the wake. It is entirely equivalent to results obtained long ago by Rayleigh (1945, §§ 296, 335), who discussed scattering by spherical and cylindrical density inhomogeneities. More recently mechanisms of this sort have been considered in the theory of jet noise by Morfey (1973), Ffowcs Williams & Howe (1975) and Howe (1975*b*).

Case (ii). *Kutta condition imposed.* The acoustic pressure, which vanishes identically when $U_s = U_1$, is given by

$$\frac{p}{\rho_2} = \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\frac{\Gamma(U_1-U_s)\sin\frac{1}{2}\theta}{\pi(1+M_2\cos\theta)R^{\frac{1}{2}}}\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right]. \tag{5.48}$$

Subtracting the direct radiation from the line vortex, the field associated with the wake can be expressed in the form

$$\frac{p}{\rho_2} = \left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)\frac{\Gamma U_s\sin\frac{1}{2}\theta}{2\pi(1+M_2\cos\theta)R^{\frac{1}{2}}}\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right] + \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\frac{\Gamma U_1\sin\frac{1}{2}\theta}{\pi(1+M_2\cos\theta)R^{\frac{1}{2}}}\left[\frac{\cos\frac{1}{2}\theta_0}{R_0^{\frac{1}{2}}}\right]. \tag{5.49}$$

The first term on the right is just the density-scattered radiation of (5.47). The second term is the additional contribution due to the imposition of the Kutta condition and arises from the vorticity shed from the edge of the plate.

In both case (i) and case (ii) the sound pressure p varies as $\rho_2 vU$, where v and U respectively characterize the velocities of the perturbed and mean flows. This is typical of the parametric dependence of the Aeolian sounds associated with two-dimensional eddies near a rigid half-plane (Ffowcs Williams 1969).

In conclusion let us note that, although we have presented no details, it is a relatively simple matter to determine the explicit form of the disturbed incompressible motion of the vortex sheet. In both of the above cases the sheet exhibits no unbounded unstable growth as a result of the passage of the vortex. This is obvious when the Kutta condition is imposed and $U_s = U_1$. In other cases, when the line vortex is far downstream of the edge, the disturbed flow in the wake is steady in a frame translating with the vortex. The absence of instabilities is a consequence of the rather good analytic properties of the vortex-induced forcing velocity $v_s(x_1, t)$ for complex values of x_1. Of course, these remarks are strictly appropriate only in the context of the present linear theory. In practice instabilities will develop because of nonlinear effects, although for a sufficiently weak incident vortex, or high mean-flow velocity, this could well occur far downstream and involve no substantial diffractively amplified acoustic radiation.

6. Conclusion

In this paper we have discussed the linear two-dimensional theory of the generation of sound during the convection of a frozen turbulent eddy past both finite and semi-infinite rigid plates, with and without the application of a Kutta condition and with and without the presence of a mean vortex sheet in the wake. It has been argued that this constitutes a more realistic model of a mean-flow/edge interaction involving a convected turbulent quadrupole than one which assumes the quadrupole to be a line source fixed relative to the plate. There are significant effects of convection. To the order of approximation to which the sound from aerodynamic sources near a scattering body is usually estimated, the imposition of a Kutta condition leads to the complete cancellation of the sound generated when frozen turbulence convects past a semi-infinite plate, and to the cancellation of the diffraction field due to the trailing edge in the case of an airfoil of compact chord. The cancellation is brought about by the shed vorticity, which smooths out large pressure gradients in the vicinity of the trailing edge.

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REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1965 *Handbook of Mathematical Functions*. Dover.
BECHERT, D. & MICHEL, U. 1975 *Acustica*, **33**, 287.
CRIGHTON, D. G. 1972 *Proc. Roy. Soc.* A **330**, 185.
CRIGHTON, D. G. 1975 *J. Fluid Mech.* **72**, 209.
CRIGHTON, D. G. & LEPPINGTON, F. G. 1970 *J. Fluid Mech.* **43**, 721.
CRIGHTON, D. G. & LEPPINGTON, F. G. 1971 *J. Fluid Mech.* **46**, 577.
CRIGHTON, D. G. & LEPPINGTON, F. G. 1974 *J. Fluid Mech.* **64**, 393.
CURLE, N. 1955 *Proc. Roy. Soc.* A **231**, 505.
DAVIES, S. S. 1975 *A.I.A.A. J.* **13**, 375.
DOWLING, A. 1976 *J. Fluid Mech.* **74**, 529.
FFOWCS WILLIAMS, J. E. 1969 *Ann. Rev. Fluid Mech.* **1**, 197.
FFOWCS WILLIAMS, J. E. 1974 *J. Fluid Mech.* **66**, 791.
FFOWCS WILLIAMS, J. E. & HALL, L. H. 1970 *J. Fluid Mech.* **40**, 657.
FFOWCS WILLIAMS, J. E. & HOWE, M. S. 1975 *J. Fluid Mech.* **70**, 605.
FFOWCS WILLIAMS, J. E. & LOVELY, J. 1975 *J. Fluid Mech.* **71**, 689.
FILOTAS, L. T. 1969 *Toronto Univ. UTIAS Rep.* no. 139.
GARABEDIAN, P. R. 1964 *Partial Differential Equations*. Wiley.
GARABEDIAN, P. R. & LIEBERSTEIN, H. M. 1958 *J. Aero. Sci.* **27**, 109.
GOLDSTEIN, M. E. 1974 *N.A.S.A. Special Publication*, no. 346.
GRAHAM, J. M. R. 1970 *Aero. Quart.* **21**, 182.
HADAMARD, J. 1952 *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Dover.
HOWE, M. S. 1967 *J. Fluid Mech.* **30**, 497.
HOWE, M. S. 1974 *Proc. Roy. Soc.* A **337**, 413.
HOWE, M. S. 1975*a* *J. Fluid Mech.* **67**, 597.
HOWE, M. S. 1975*b* *J. Fluid Mech.* **71**, 625.
JONES, D. S. 1972 *J. Inst. Math. Appl.* **9**, 114.
JONES, D. S. & MORGAN, J. D. 1972 *Proc. Camb. Phil. Soc.* **72**, 465.
JONES, D. S. & MORGAN, J. D. 1974 *Proc. Roy. Soc.* A **338**, 17.
KÁRMÁN, T. VON & SEARS, W. R. 1938 *J. Aero. Sci.* **5**, 397.
LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.
LEVINE, H. 1975 *Philips Res. Rep.* **30**, 240.
LIEBERSTEIN, H. M. 1959 *MRC Tech. Summ. Rep.* no. 81. Madison, Wisconsin.
LIGHTHILL, M. J. 1952 *Proc. Roy. Soc.* A **221**, 564.
LIGHTHILL, M. J. 1960 *Phil. Trans.* A **252**, 397.
MÖHRING, W. 1975 *J. Sound Vib.* **38**, 403.
MORFEY, C. L. 1973 *J. Sound Vib.* **31**, 391.
MORGAN, J. D. 1974 *Quart. J. Mech. Appl. Math.* **27**, 465.
MUGRIDGE, B. D. 1971 *Aero. Quart.* **22**, 301.
MORSE, P. M. & INGARD, K. U. 1968 *Theoretical Acoustics*. McGraw Hill.
NOBLE, B. 1958 *Methods Based on the Wiener–Hopf Technique*. Pergamon.
ORSZAG, S. A. & CROW, S. C. 1970 *Stud. Appl. Math.* **49**, 167.
RAYLEIGH, LORD 1945 *Theory of Sound*. Dover.